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18.034 Honors Differential Equations  
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## LECTURE 7. MECHANICAL OSCILLATION

**The spring-mass system and the electric circuit.** If a mass  $m$  is attached to one end of a suspended spring, it will produce an elongation, say  $y_0$ , which, according to Hooke's law, is proportional to the force of gravity

$$ky_0 = mg,$$

where  $k > 0$  is the stiffness constant of spring and  $g > 0$  is the gravitational acceleration. If an additional force is applied and the spring is stretched and if the force is then removed, the spring-mass system will start oscillating. The problem is to determine the subsequent motion.

Let  $y = y(t)$  be the displacement of the mass from the rest state  $y_0$ . The forces exerted on the mass are the force of gravity plus the tension in the spring  $k(y_0 + y)$ , which acts in the opposite direction to the exerted force. Thus,

$$F = mg - k(y_0 + y) = -ky.$$

By Newton's second law of motion we obtain the equation of motion

$$my'' + ky = 0.$$

In a more complete description of the motion of the spring, we take into account of the damping effect and the external force. If the mass is in a resisting medium in which the damping is proportional to the velocity  $y'$ , and if an external force  $f(t)$  is exerted then the equation of motion becomes

$$(7.1) \quad my'' + ry' + ky = f(t),$$

where  $r > 0$  is the constant of resistance.

Next, we explore a striking analogy between the spring-mass system and an RLC circuit. Let a source of electromotive force be connected in a series with a capacitor, an inductor, and a resistor. Let  $v = v(t)$  measure the voltage across the capacitor. By Ohm's law, the voltage equation is

$$(7.2) \quad Lv'' + Rv' + \frac{1}{C}v = \frac{1}{C}v_0(t),$$

where  $L > 0$  is the inductance,  $R > 0$  is the resistance,  $C > 0$  is the capacitance,  $1/C$  is the elastance, and  $v_0(t)$  is the voltage at the source.

Both (7.1) and (7.2) are linear second-order differential equations with constant coefficients. Analysis of these equations leads to the concepts of phase lag and gain, beats and resonance, which are subjects of this lecture.

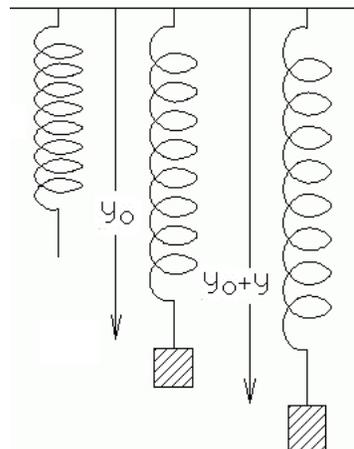


Figure 7.1. The spring-mass system

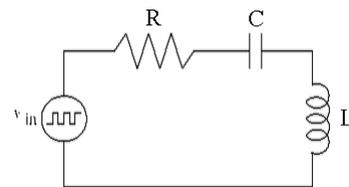


Figure 7.2. The RLC circuit

**Free oscillations.** We first consider the undamped unforced oscillation

$$(7.3) \quad my'' + ky = 0 \quad \text{or} \quad y'' + \omega_0^2 y = 0$$

where  $\omega_0^2 = m/k > 0$  is called the *natural frequency*.

From the last lecture, the general solution of (7.3) is given by

$$y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

where  $c_1$  and  $c_2$  are arbitrary constants. Further, the solution can be written as

$$y(t) = A \cos(\omega_0 t - \phi),$$

and it describes an oscillatory motion, called the *simple harmonic motion*. Here,  $A \geq 0$  (or better  $|A|$ ) is called the *amplitude*, and  $\phi$  is called the *phase lag*. The motion is periodic with the period  $2\pi/\omega_0$ .

**Exercise.** (Conservation of energy) Let  $v = y'$ . Show that

$$\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E$$

defines the solution of (7.3) implicitly. The equation expresses the principle of *conservation of energy*;  $(1/2)mv^2$  represents the kinetic energy of the moving mass ( $v$  is the velocity) and  $(1/2)ky^2$  represents the work

$$W = \int_0^y F dy = \int_0^y ky dy$$

needed to stretch the spring by the displacement  $y$ , and hence the potential energy. In the absence of damping or external forcing, one would expect that the total energy, the kinetic plus the potential, remain constant in time. This exercise confirms the expectation.

**Undamped forced oscillations.** Next, we consider the undamped oscillation with forcing

$$(7.4) \quad y'' + \omega_0^2 y = \omega_0^2 f(t), \quad \omega_0^2 > 0.$$

In this lecture, the forcing term is taken to be a sinusoidal input

$$(7.5) \quad f(t) = A \cos \omega t \quad \text{or} \quad f(t) = B \sin \omega t,$$

where  $A, B$  and  $\omega > 0$  are real. Our choice of input functions is partly supported by that the output of an alternating-current generator is, as a rule, sinusoidal. But, more important is that an arbitrary periodic input is approximated by sums of trigonometric polynomials

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi}{L} kt + b_k \sin \frac{2\pi}{L} kt \right),$$

called the Fourier series. If we know how a system responds to the particular inputs of the form in (7.5) then we can treat an arbitrary periodic input by the general principle of superposition, with the help of Fourier analysis.

Let  $\omega \neq \omega_0$ , and we the trial solution

$$y(t) = a \cos \omega t \quad \text{or} \quad y(t) = b \sin \omega t$$

of (7.4) with (7.5) leads to

$$a = \frac{\omega_0}{\omega_0^2 - \omega^2} A, \quad b = \frac{\omega_0}{\omega_0^2 - \omega^2} B.$$

The ratio  $a/A$  or  $b/B$  is called the *gain* of the system. Note that the closer  $\omega$  is to  $\omega_0$  the larger the gain becomes.

*Beats.* Assuming that  $\omega \neq \omega_0$ , we consider

$$y'' + \omega_0^2 y = (\omega^2 - \omega_0^2) \cos \omega t.$$

The general solution is given by  $y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t - \cos \omega t$ .

Prescribed the null initial conditions  $y(0) = y'(0) = 0$ , the corresponding rest solution is

$$y(t) = \cos \omega_0 t - \cos \omega t = 2 \sin \left( \frac{\omega + \omega_0}{2} t \right) \sin \left( \frac{\omega - \omega_0}{2} t \right).$$

If  $\omega$  is close to  $\omega_0$  and if both are large, then the second factor is slowly varying compared to the first, as is shown in the figure below.

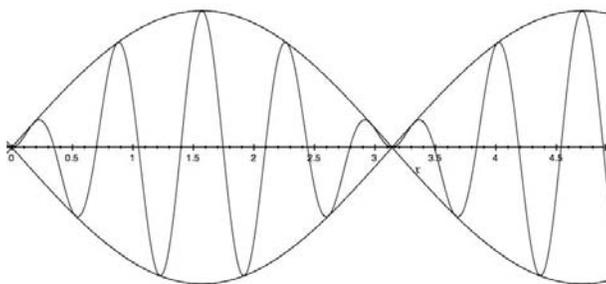


Figure 7.3. Beats.

The low-frequency factor  $\sin \left( \frac{\omega - \omega_0}{2} t \right)$ , in this sense, is viewed to induce amplitude modulation on the high-frequency factor  $\sin \left( \frac{\omega + \omega_0}{2} t \right)$ . Such a modulated oscillation is called *beats*.

For a general initial data, we study the complex exponential

$$y(t) = Ae^{i\omega t} + Be^{i(1+\epsilon)\omega t} = e^{i\omega t}(A + Be^{i\epsilon\omega t}).$$

The behavior of this function is sketched in the figure below.

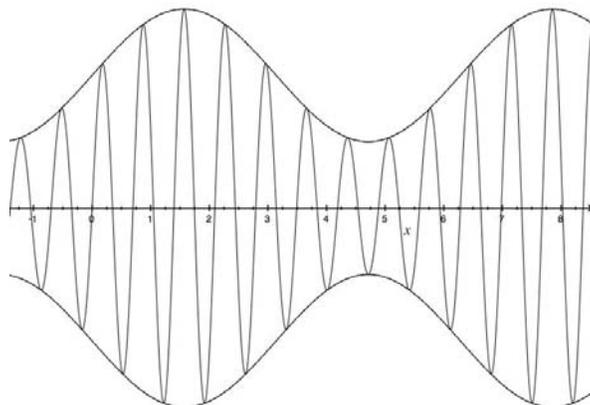


Figure 7.4.  $A = 1, B = 0.5, \omega = 1, \epsilon = 0.1$ .

The function  $|y(t)| = \sqrt{A^2 + B^2 + 2AB \cos(\epsilon\omega t)}$  serves as the amplitude modulation and it shows the “beats” effect. The maximum amplitude is  $A + B$  at time  $t = 2k\pi/\epsilon\omega$ . It occurs when the

phases of two constituent waves are lined up so that they reinforce. The minimum amplitude is  $|A-B|$  ( $\neq 0$ ) and it occurs when the two waves are perfectly out of sync and experience destructive interference.

Beats have applications in tuning musical instruments or radar technology. If a tuning instrument and a piano string sound simultaneously at nearly the same frequency, the beats allow you to tell how far apart the two notes are even. The slower the beats, the more nearly the string is in tune.

In 1902 the heterodyne method was invented in which a circuit at the receiver generates a signal that differs only slightly in frequency from the high-frequency signal of the sender. The difference produces a frequency in the audible range. As a spin-off from radar technology, in 1946, the heterodyne method was used to detect one-centimeter microwave radiation from the sun. This experiment marked as a major step in radio astronomy and microwave spectroscopy.

*Resonance.* When  $\omega = \omega_0$ , the system experiences *resonance*. We consider

$$y'' + \omega_0 y = \omega_0^2 A \cos \omega_0 t \quad \text{or} \quad y'' + \omega_0^2 y = \omega_0^2 B \sin \omega_0 t, \quad \omega_0 > 0.$$

The particular solutions are

$$y(t) = \frac{1}{2} A \omega_0 t \sin \omega_0 t, \quad y(t) = -\frac{1}{2} B \omega_0 t \cos \omega_0 t,$$

respectively. The amplitude  $(1/2)A\omega_0 t$  or  $(1/2)B\omega_0 t$  is proportional to  $t$  and is unbounded as  $t \rightarrow \infty$ . See the figure below.

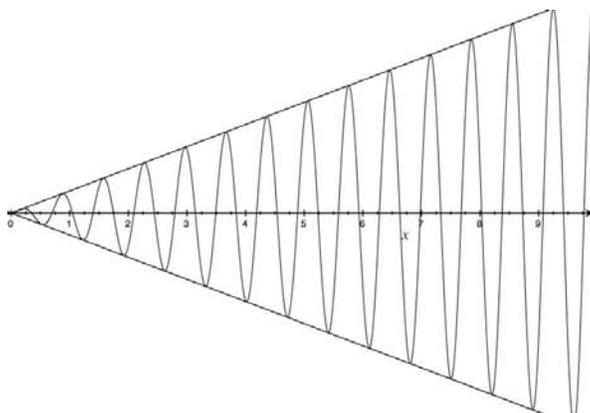


Figure 7.5. Resonance.

The phenomenon of resonance has profound importance in engineering. The failure of the Tacoma bridge was explained by some authorities on the basis of forced vibrations.

When the trumpets sounded, the people shouted, and at the sound of the trumpet, when the people gave a loud shout, the wall collapsed; so every man charged straight in, and they took the city. (Joshua 6:20)