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## UNIT II: SECOND-ORDER LINEAR EQUATIONS

We discuss the existence and uniqueness for second-order linear differential equations with constant coefficients by means of general principles which are valid for equations with variable coefficients. We develop techniques of the Wronskian and apply to study the oscillatory behavior. We also give qualitative results which depend on conditions at an interior maximum and minimum and apply to second-order equations.

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### LECTURE 6. SECOND-ORDER LINEAR EQUATIONS

The most intensively studied ordinary differential equations are second-order linear equations of the form

$$(6.1) \quad p_0(t)y'' + p_1(t)y' + p_2(t)y = f(t),$$

where  $' = \frac{d}{dt}$  is differentiation in the  $t$ -variable, and  $p_j(t), f(t), j = 0, 1, 2$ , are continuous functions on an interval  $I$ .

*Remark 6.1* (Remark on singular points). If  $p_0(t)$  vanishes at some point, say  $t_0$ , then we say that (6.1) has a *singular point* at  $t_0$ . For example, the Legendre equation

$$((1 - t^2)y')' + \lambda y = 0 \quad \text{or} \quad (1 - t^2)y'' - 2ty' + \lambda y = 0$$

has singular points at  $t = \pm 1$ . While the equation admits polynomial solutions, called the Legendre polynomials, for  $\lambda = n(n + 1)$ , other nontrivial solutions develop a singularity at either  $t = 1$  or  $t = -1$ .

We do not treat singular points in this course.

**Bases of solutions.** If  $p_0(t) \neq 0$  on  $I$  then (6.1) reduces to the normal form

$$(6.2) \quad y'' + p(t)y' + q(t)y = f(t),$$

where  $p(t), q(t)$  and  $f(t)$  are continuous on  $I$ .

We consider the corresponding homogeneous equation

$$(6.3) \quad y'' + p(t)y' + q(t)y = 0.$$

Since the equation is linear, the principle of superposition applies, and if  $\phi$  and  $\psi$  are solutions of (6.3) then so is their *linear combination*  $c_1\phi + c_2\psi$  for any constants  $c_1, c_2$ .

**Definition 6.2.** On a given interval two functions are said *linearly dependent* if one of them is a constant multiple of the other. They are said *linearly independent*, otherwise.

For example,  $e^{-t}$  and  $e^{-2t}$  are linearly independent on every interval of  $t$ , although they agree at  $t = 0$ .

If  $\phi$  and  $\psi$  are solutions of (6.3) and, in addition, if  $\phi$  and  $\psi$  are linearly independent then every solution of (6.3) is of the form  $c_1\phi + c_2\psi$  for some constants  $c_1, c_2$ . We call such a pair of solutions  $\{\phi, \psi\}$  a *basis of solutions* of (6.3). The existence of a basis of solutions is a fundamental result for (6.3) and for a broader class of linear homogeneous differential equations, and we will prove it later.

**Constant-coefficient differential equations.** We construct a basis of solution of a second-order *constant-coefficient* differential equation

$$(6.4) \quad y'' + py' + qy = 0,$$

where  $p, q$  are constants.

The trick is to set  $y(t) = e^{\lambda t}$ , where  $\lambda \in \mathbb{C}$ . We use  $(e^{\lambda t})' = \lambda e^{\lambda t}$  and  $(e^{\lambda t})'' = \lambda^2 e^{\lambda t}$  to write (6.4) as

$$e^{\lambda t}(\lambda^2 + p\lambda + q) = 0.$$

Since  $e^{\lambda t} \neq 0$ , it follows that  $y(t) = e^{\lambda t}$  is a solution of (6.4) if and only if  $\lambda$  is a root of the quadratic polynomial

$$(6.5) \quad p(\lambda) = \lambda^2 + p\lambda + q,$$

called the *characteristic polynomial* of (6.4). The characteristic polynomial has two roots (counting multiplicity)

$$\lambda = \frac{-p \pm \sqrt{\Delta}}{2}, \quad \Delta = p^2 - 4q.$$

We divide our discussion according to the nature of the roots, which is determined by the sign of the *discriminant*  $\Delta$ .

If  $\Delta > 0$  then both roots are real, and

$$e^{(-p+\sqrt{\Delta})t/2}, \quad e^{(-p-\sqrt{\Delta})t/2}$$

are two solutions of (6.4). Moreover, they are linearly independent. Therefore, they form a basis of solutions of (6.4).

If  $\Delta = 0$  then the method yields only the single solution  $e^{-pt/2}$ . Using the method suggested by Lagrange, we try  $ve^{-pt/2}$  as a second solution, where  $v = v(t)$ . Upon substitution, (6.4) becomes

$$(ve^{-pt/2})'' + p(ve^{-pt/2})' + qve^{-pt/2} = v''e^{-pt/2} = 0.$$

Thus,  $v'' = 0$  must hold and we obtain the family of solutions  $y(t) = (c_1 + c_2t)e^{-pt/2}$ . Since

$$e^{-pt/2}, \quad te^{-pt/2}$$

are linearly independent, they form a basis of solutions of (6.4).

If  $\Delta < 0$ , then both roots are complex, and

$$e^{-pt/2+i\sqrt{-\Delta}t/2}, \quad e^{-pt/2-i\sqrt{-\Delta}t/2}$$

are two (complex) solutions of (6.4). Then, we infer that

$$\frac{u+v}{2} = e^{-pt/2} \cos(\sqrt{-\Delta}t/2), \quad \frac{u-v}{2i} = e^{-pt/2} \sin(\sqrt{-\Delta}t/2)$$

are two real solutions of (6.4). Since they are linearly independent, they form a basis of (real) solutions of (6.4).

The *principle of equating real parts* holds for, more generally, variable-coefficient equations, and if  $z(t) = x(t) + iy(t)$  is a complex solution of

$$z'' + p(t)z' + q(t)z = 0,$$

where  $p(t)$  and  $q(t)$  are real-valued functions then so are  $x$  and  $y$ .

**Exercise.** Prove the principle of equating real parts.

**Inhomogeneous equations and initial value problems.** Next, we turn to the inhomogeneous equation (6.2) with a general  $f(t)$ . The following result enables us to get a family of solutions of (6.2) from a particular solution.

**Principle of the Complementary Solution.** If  $y_p$  is a solution of the inhomogeneous equation (6.2), called a *particular solution* and if  $\phi$  and  $\psi$  form a basis of solutions of the homogeneous equation (6.3), then the general solution of (6.2) is given by  $y_p + c_1\phi + c_2\psi$ , where  $c_1$  and  $c_2$  are arbitrary constants.

*Proof.* Let  $y$  be a solution of (6.2). Since  $y - y_p$  satisfies the homogeneous equation (6.3), it should take the form  $c_1\phi + c_2\psi$  for some constants  $c_1, c_2$ .  $\square$

**Example 6.3.** Let us consider

$$(6.6) \quad y'' + y = 3 \sin 2t.$$

We try  $y_p(t) = A \sin 2t$ , where  $A$  is a constant, for a particular solution. Since  $y'' = -4A \sin 2t$ , the differential equation becomes

$$(-4A + A) \sin 2t = 3 \sin 2t.$$

Thus,  $A = -1$ , and  $y_p(t) = -\sin 2t$  is a particular solution of (6.6). On the other hand,  $\cos t$  and  $\sin t$  form a basis of solutions of the corresponding homogeneous equation  $y'' + y = 0$ . Therefore, the general solution of (6.6) is given

$$y(t) = -\sin 2t + c_1 \cos t + c_2 \sin t.$$

Particular solutions with polynomial forcing terms are treated similarly. As an illustration, we consider

$$y'' + py' + qy = ct + d,$$

where  $p, q, c, d$  are constants. We first look for a particular solution of the form  $y_p(t) = at + b$ . Upon substitution, we obtain the equations

$$qa = c, \quad pa + qb = 0.$$

Unless  $q = 0$ , these give the particular solution

$$y_p(t) = \frac{c}{q}t + \frac{qd - pc}{q^2}.$$

If  $q = 0$  but  $p \neq 0$ , then we look for a particular solution as a quadratic polynomial;  $y'' + y' = x$  has a solution  $y_p(t) = t^2/2 - t$ . Finally, when  $p = q = 0$ , the equation  $y'' = ct + d$  has a cubic solution  $y_p(t) = ct^3/6 + dt^2/2$ .

Other methods to construct a particular solution, such as variation of parameters and the annihilator method, will be discussed later for a more general class of differential equations.

The equation (6.2) is sometimes supplemented with additional initial or boundary conditions.

**Example 6.4.** We illustrate how to solve initial value problem for differential equations in the form (6.2) by considering (6.6) with the initial conditions

$$y(0) = y'(0) = 0.$$

We have shown that the general solution of (6.6) is given as

$$y(t) = -\sin 2t + c_1 \cos t + c_2 \sin t,$$

where  $c_1$  and  $c_2$  are constants. Such a function will satisfy  $y(0) = 0$  if and only if  $c_1 = 0$ , so that  $y'(t) = -2 \cos 2t + c_2 \cos t$ . In particular,  $y'(0) = -2 + c_2$ . Hence the function  $y(t) = \sin 2t - 2 \sin t$  solves the stated initial value problem.

**Historical introduction to linear differential equations.** The history of differential equations began in the 17th century when Newton, Leibniz, and the Bernoullis solved some simple differential equations of first- and second-order arising from problems in geometry and mechanics. These early discoveries seemed to suggest that the solutions of *all* differential equations based on geometric and physical problems could be expressed in terms of the familiar elementary functions of calculus. Therefore, much of the early work was devoted to developing ingenious techniques for solving differential equations by addition, subtraction, multiplication, division, composition, and integration, applied only finitely many times to the familiar functions of calculus.

But, it soon became apparent that relatively few differential equations could be solved by elementary means. Little by little, mathematicians began to realize that it was hopeless to try to discover methods for solving all differential equations. Instead, they found it more fruitful to ask whether or not a given differential equation has any solution at all and, when it has, to try to deduce properties of the solution from the differential equation itself. Within this framework, mathematicians began to think of differential equations as new sources of functions.

In the 1820's, Cauchy obtained the first "existence theorem" for differential equations. He proved that every first-order equation of the form  $y' = f(x, y)$  has a solution whenever the right member,  $f(x, y)$ , satisfies certain general conditions. One important example is the Riccati equation

$$y' = p(x)y^2 + q(x)y + r(x),$$

where  $p, q$  and  $r$  are given functions. Cauchy's work implies the existence of a solution of the Riccati equation in any open interval  $(-a, a)$  about the origin, provided  $p, q$  and  $r$  have power-series expansions in  $(-a, a)$ . In 1841 Joseph Liouville showed that in some cases this solution cannot be obtained by elementary means.

Experiences has shown that it is difficult to obtain results of much generality about solutions of differential equations, except for a few types. Among these are the so-called *linear* differential equations, which occur in a great variety of scientific problems. We develop the principal results concerning these equations.