18.034 Honors Differential Equations
Spring 2009

LECTURE 5. LINEAR FRACTIONAL EQUATIONS AND SUBSTITUTION

Homogeneous equations. A function $f(x,y)$ is said *homogeneous of degree* m if

$$f(\lambda x, \lambda y) = \lambda^m f(x,y), \qquad \lambda > 0.$$

If P and Q are homogeneous of the same degree then P/Q is homogeneous of degree zero. Indeed,

$$\frac{P(\lambda x, \lambda y)}{Q(\lambda x, \lambda y)} = \frac{\lambda^m P(x,y)}{\lambda^m Q(x,y)} = \frac{P(x,y)}{Q(x,y)}$$

Let $g = P/Q$. Taking $\lambda = 1/x$, $x > 0$,

$$g(x,y) = g(\lambda x, \lambda y) = g\left(1, \frac{y}{x}\right) =: f\left(\frac{y}{x}\right).$$

Therefore, under the conditions the differential equation

$$P(x,y)dx = Q(x,y)dy$$

is equivalent to

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{5.1}$$

These equations are called *homogeneous*.

Theorem 5.1. *Let $y = \phi(x)$ satisfy* (5.1) *on an interval I.*

(i) *If $c \neq 0$ the function $z = (1/c)\phi(cx)$ satisfies $z' = f(z/x)$ on the corresponding interval.*
(ii) *If $x \neq 0$ and $f(v) \neq v$ then under the change of variables $v = y/x$, the equation* (5.1) *becomes*

$$\frac{dv}{f(v) - v} = \frac{dx}{x} = d(\log x). \tag{5.2}$$

The proof is left as an exercise.
The equation (5.2) is separable, and its solution is given by $x = k\exp(\int dv/(f(v) - v))$.

We now use the result to study an important class of differential equations, called *linear fractional equations*

$$(ax + by)y' - (cx + dy) = 0, \qquad ad - bc \neq 0, \tag{5.3}$$

or

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}. \tag{5.3′}$$

The equation (5.3′) is of the form (5.1) with $f(v) = \dfrac{c + dv}{a + bv}$. The condition $ad - bc \neq 0$ ensures that $f(v) \neq v$. Hence, the results in Theorem 5.1 apply.

Upon the substitution $v = y/x$, the equation (5.3′) becomes

$$x\frac{dv}{dx} + v = \frac{c + dv}{a + bv}.$$

Separating variables, we further write it as

$$\frac{(a + bv)dv}{bv^2 + (a - d)v - c} + \frac{dx}{x} = 0.$$

The solution, then, is given as

$$x = k \exp\left(-\int \frac{a+bv}{bv^2+(a-d)v-c}dv\right),$$

where $v = y/x$ and k is a constant.

Invariant radii. Alternatively, we can study a homogeneous equation (5.1) in polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Let γ be the angle that the tangent direction of a solution curve and let $\psi = \gamma - \theta$. Then,

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\psi = \frac{\cot\gamma\cot\theta + 1}{\cot\theta - \cot\gamma}.$$

Since $\gamma = y' = f(y/x) = f(\tan\theta)$, it follows that

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1+\tan\gamma\tan\theta}{\tan\gamma - \tan\theta} = \frac{1+(\tan\theta)f(\tan\theta)}{f(\tan\theta)-\tan\theta} := Q(\theta).$$

By integrating, we obtain the solution of (5.1) as

$$r(\theta) = r(0)\exp\int_0^\theta Q(\theta)d\theta.$$

The function on the right side is well-defined, as long as $\tan\gamma \neq \tan\theta$, that is, as long as $y' \neq y/x$.

If a solution curve $r(\theta)$ is such that the denominator of $Q(\theta)$ vanishes along it then (5.1) is equivalent to $d\theta/dr = 0$. Hence, these radii are particular solution curve of (5.1). They are called *invariant radii*. They are the solutions $y = (\tan\theta)x$ for which $y' = \tan\theta = f(\tan\theta)$.

For the linear fractional equation (5.3′) the slopes of invariant radii are solutions of

$$\tan\theta = \frac{c + d\tan\theta}{a + b\tan\theta}.$$

They are the roots of the quadratic equation

$$b(\tan\theta)^2 + (a-d)(\tan\theta) - c = 0.$$

If $r = g(\theta)$ is a solution of (5.1), then so is $r = \lambda g(\theta)$. Indeed, the solution curve is invariant under the similar transformation $(x, y) \mapsto (\lambda x, \lambda y)$. To interpret, in the sector between any two adjacent invariant radii are all "geometrically similar". This fact is useful in sketching the solution curves.

We discuss another example of homogeneous equations.

Example 5.2. If $xy \neq x^2$ then the equation $(xy - x^2)y' = y^2$ can be written as

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} = \frac{(y/x)^2}{(y/x)-1}.$$

That is, the equation is of the form (5.1) with $f(v) = \dfrac{v^2}{v-1}$. The equation $f(v) = v$ has only one solution $v = 0$, which corresponds to $y = 0$. If $xv \neq 0$, then the solution is given as $v = \log|kxv|$, where k is a constant. Dropping the absolute value,

$$y = x\log(ky) \quad \text{or} \quad ky = cx\log(ky)$$

Some second-order equations. Many second-order differential equations can be reduced to first-order equations by appropriate substitution.

We first consider a second-order equation of the form

$$y'' = f(x, y'). \tag{5.4}$$

That is, the differential equation does not involve the dependent variable y explicitly. Let $v = y'$. Since $y'' = (y')' = v' = dv/dx$, the equation (5.4) is reduced to the first-order equation

$$\frac{dv}{dx} = f(x, v).$$

If this equation can be solved for v then y can be obtained by integrating $\dfrac{dy}{dx} = v$.

Exercises. Solve the differential equation:
1. $x^2y'' + 2xy' - 1 = 0$ 2. $y'' + x(y')^2 = 0$ 3. $x^2y'' = (y')^2, \quad x > 0$.

Next, we consider a second-order equation of the form

$$y'' = g(y, y'). \tag{5.5}$$

That is, the differential equation does not involve the independent variable x explicitly.

Let $v = y'$, and we express (5.5) as a first-order differential equation with y as the independent variable. By the chain rule,

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v.$$

Using this in (5.5) we obtain

$$v\frac{dv}{dy} = g(y, v).$$

Provided that this first-order equation can be solved, we obtain v as a function of y. Then, y can be obtained by solving $\dfrac{dy}{dx} = v(y)$.

The expression vdv/dy is more subtle than dv/dx, since y is thought of as a function of x in the differential equation and is treated as as independent variable in the chain rule. We include a mathematical justification, for completeness*. The idea is that if f is differentiable function then $df(u)$ has the same form no matter what u is an independent variable or is a differentiable function $\phi(x)$ of another variable x. Namely,

$$df(\phi(x)) = f'(\phi(x))\phi'(x)dx = f'(u)du.$$

Let $y = \phi(x)$ satisfies (5.5). Then,

$$\phi''(x) = g(\phi(x), \phi'(x)). \tag{5.6}$$

With $v = \phi'(x)$ and $dv = \phi''(x)dx$, the equation in differentiable form $vdv = g(y, v)dy$ becomes

$$\phi'(x)\phi''(x)dx = g(\phi(x), \phi'(x))\phi'(x)dx, \tag{5.7}$$

which is different from (5.6) by the factor of $\phi'(x)dx$. Therefore, (5.5) is equivalent to $vdv = g(y, v)dy$, provided that $\phi'(x) \neq 0$. (If the zeros of $\phi'(x)$ are isolated, and (5.6) is equivalent to (5.7) except these isolated zeros. In this case, by continuity, (5.6) holds everywhere.)

Exercises. Solve the differential equation:
1. $yy'' + (y')^2 = 0$ 2. $y'' + y(y')^3 = 0$ 3. $2y^2y'' + 2y(y')^2 = 1$.

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*It requires rudimentary knowledge on differentiable forms.