18.034 Honors Differential Equations
Spring 2009

## LECTURE 4. SEPARABLE EQUATIONS

**Separable equations.** Separable equations are differential equations of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}. \tag{4.1}$$

For example, $x + yy' = 0$ and $y' = y^2 - 1$. A separable equation (4.1) can be written in the differential form as

$$f(x)dx = g(y)dy. \tag{4.2}$$

Then, it can be solved formally by integrating both sides of (4.2).

We state and prove the rigorous theory of local solutions for (4.2) (and hence (4.1)).

**Theorem 4.1.** *Let $f(x)$ and $g(x)$ be continuous in the rectangle $R = \{(x, y) : a < x < b, c < y < d\}$. In addition, if $f$ and $g$ do not vanish simultaneously at any point of $R$, then* (4.2) *has one and only one solution through each point $(x_0, y_0) \in R$. The solution is given by*

$$\int_{x_0}^{x} f(x)dx = \int_{y_0}^{y} g(y)dy \tag{4.3}$$

It is essential that $f$ and $g$ do not vanish simultaneously. For example, $xdx = -ydy$ has no solution through the origin.

*Proof.* Note that $f(x) \neq 0$ for $a < x < b$ or $g(y) \neq 0$ for $c < y < d$. Without loss of generality, we assume $g(y) > 0$ for $c < y < d$.

Let

$$F(x) = \int_{x_0}^{x} f(x)dx, \qquad G(x) = \int_{y_0}^{y} g(y)dy$$

so that (4.3) becomes

$$F(x) = G(x). \tag{4.4}$$

Since $G'(y) = g(y) > 0$ in $R$, by the inverse function theorem, $G^{-1}$ exists and (4.4) can be written as $y = G^{-1}(F(x))$. That means, $\frac{dy}{dx}$ exists. Then, by differentiating (4.4), we get

$$F'(x) = G'(x)\frac{dy}{dx}, \quad \text{or} \quad f(x) = g(y)\frac{dy}{dx}.$$

This implies (4.2). Moreover, (4.3) gives the initial condition that $y = y_0$ when $x = x_0$.

To prove the uniqueness, let $y$ be one solution of (4.2) and $z$ be another solution with the same initial condition. Under the hypothesis, the equation

$$\frac{dz}{dx} = \frac{f(x)}{g(z)}$$

implies that $dz/dx$ exists for any $(x, z) \in R$. Let

$$u = G(y), \qquad v = G(z).$$

Then,

$$\frac{du}{dx} = G'(y)\frac{dy}{dx} = g(y)\frac{dy}{dx} = f(x).$$

Similarly, $\frac{dz}{dx} = f(x)$. Since $u$ and $v$ have the same derivative, they differ by a constant. On the other hand, the initial conditions for $u$ and $v$ at $x_0$ agree. Therefore, $u = v$ everywhere in $R$. This completes the proof. □

**Example 4.2.** Consider the initial value problem

$$\frac{dy}{dx} = 1 + y^2, \qquad y(0) = 1. \tag{4.5}$$

Separating the variables, we write the differential equation as

$$\frac{dy}{1+y^2} = dx.$$

Since the constant function never vanishes, upon integration and evaluation, we obtain

$$\tan^{-1} y = x + c, \qquad \tan^{-1} 1 = c.$$

Therefore, the (unique) solution of (4.5) is $y = \tan(x + \pi/4)$. The same result is obtained by integrating between corresponding limits

$$\int_1^y \frac{dy}{1+y^2} = \int_0^x dx.$$

**Orthogonal trajectories.** If two families of curves are such that every curve of one family intersects the curves of the other family at a right angle, then we say that the two families are *orthogonal trajectories* of each other. For example, the coordinate lines:

$$x = c_1, \qquad y = c_2$$

in a Cartesian coordinate system form a set of orthogonal trajectories. Another example is the circles and radial lines

$$r = c_1, \qquad \theta = c_2$$

in a polar coordinate system.

Suppose a curve in the $(x, y)$-plane is such that the tangent at a point $(x, y)$ on it makes an angle $\phi$ with the $x$-axis. The orthogonal trajectory through the same point $(x, y)$ then makes an angle $\phi + \pi/2$ with the $x$-axis. Since

$$\tan(\phi + \pi/2) = -\cot\phi = -\frac{1}{\tan\phi}$$

and since the slope of the curve is $\frac{dy}{dx} = \tan\phi$, we should replace $\frac{dx}{dy}$ by $-\frac{dx}{dy}$ in the differential equation for the original family to get the differential equation for the orthogonal trajectories.

**Example 4.3.** We consider the family of circles

$$x^2 + y^2 = cx \tag{4.6}$$

tangent to the $y$ axis.

By differentiating (4.6) and by eliminating $c$, we obtain a differential equation

$$x^2 + y^2 = 2x + 2xy\frac{dy}{dx}, \quad \text{or} \quad y^2 - x^2 = 2xy\frac{dy}{dx}$$

that the family of curves (4.6) satisfies. Replace $dy/dx$ by $-dx/dy$ we get the equation of the orthogonal trajectories

$$y^2 - x^2 = -2xy\frac{dx}{dy}.$$

We write it in differential form as

$$2xydx - x^2dy + y^2dy = 0.$$

Multiplying by $1/y^2$ then gives[*]

$$d\left(\frac{x^2}{y}\right) + dy = 0,$$

and hence, $\dfrac{x^2}{y} + y = c$. We arrange it into

$$x^2 + y^2 = cy,$$

which represents a family of circles tangent to the $x$-axis.

Although the analytical steps require $x \neq 0,\ y \neq 0,$ and $y^2 \neq x^2$, the final result is valid without these restrictions.

The quantity $c$ defined by $x^2 + y^2 = cx$ is constant on the original curves of the family, but not on the orthogonal trajectories. That is why $c$ must be eliminated in the first step.

**Exercise.** Show that the orthogonal trajectories of the family of geometrically similar, coaxial ellipses

$$x^2 + my^2 = c, \qquad m > 0$$

are given by $y = \pm|x|^m$.

**Exercise.** Show that the solution curves of any separable equation $y' = f(x)g(y)$ have as orthogonal trajectories the solution curves of the separable equation $y' = -1/f(x)g(y)$.

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[*]This procedure makes the equation *exact* and the solution is defined implicitly. The factor $1/y^2$ is called a *integrating factor*. We will study exact differential equation more systematically later.