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18.034 Honors Differential Equations
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LECTURE 3. FIRST-ORDER LINEAR EQUATIONS

First-order linear differential equations. We will give a systematic method of solving first-order differential equations (of normal form)

$$(3.1) \quad y' + p(x)y = f(x)$$

on a given interval I , where p, f are continuous functions.

First, the *homogeneous* equation

$$(3.2) \quad y' + p(x)y = 0$$

is solved *by quadrature*. Let $P(x) = \int p(x)dx$ be an indefinite integral of $p(x)$. Then,

$$\frac{d}{dx}(e^{P(x)}y) = e^{P(x)}y' + p(x)e^{P(x)}y = e^{P(x)}(y' + p(x)y) = 0$$

if and only if y solves (3.2), since $e^{P(x)} \neq 0$.

Theorem 3.1. Let $p(x)$ be continuous on an interval I and $P(x) = \int p(x)dx$ be an anti-derivative of $p(x)$. Then,

$$\phi(x) = ce^{-P(x)} = ce^{-\int p(x)dx}$$

is a solution of (3.2) for any constant c . Conversely, all solutions of (3.2) are of this form.

Exercise. Show that a solution $y(x)$ of (3.2), where $p(x)$ is continuous, is: either $y(x) = 0$ for all x or $y(x) \neq 0$ for all x .

Next, we treat the *inhomogeneous* equation

$$(3.3) \quad y' + p(x)y = f(x)$$

by *variation of parameters*. Let $P(x) = \int p(x)dx$ as before. Then,

$$\frac{d}{dx}(e^{P(x)}y) = e^{P(x)}(y' + p(x)y) = e^{P(x)}f(x).$$

Hence, $ye^{P(x)} = y_0 + \int_{x_0}^x e^{P(s)}f(s)ds$ for some y_0 and for some x_0 .

Theorem 3.2. Let $p(x)$ be as in Theorem 3.1 and let $f(x)$ be continuous on an interval $x_0 \in I$. Then, the general solution of (3.3) is given by

$$y(x) = y_0e^{-P(x)} + e^{-P(x)} \int_{x_0}^x e^{P(s)}f(s)ds.$$

Moreover, $y(x_0) = y_0$ if and only if $P(x) = \int_{x_0}^x p(s)ds$.

Example 3.3. Consider the differential equation

$$(3.4) \quad y' + y = x + 3.$$

By trying $y(x) = ax + b$ one easily finds a solution $y = x + 2$ of (3.4). If $y = \phi(x)$ is another solution of (3.4), then $z = y - (x + 2)$ must satisfy the corresponding homogeneous equation $z' + z = 0$. By Theorem 3.2, then, $z = ce^{-x}$. Therefore, the general solution of (3.4) is

$$y = ce^{-x} + x + 2.$$

Exercise. (Bernoulli equations) 1. If p and q are continuous functions of x and $n \neq 1$ is constant, show that the *Bernoulli equation*

$$y' + p(x)y = q(x)y^n, \quad y > 0$$

can be reduced to a linear equation upon substitution $y = u^m$ for a suitable constant m . The condition $y > 0$ ensures that the function $u = y^{1/m}$ is meaningful. When $n = 1$ the original equation is already linear.

2. Solve the differential equation $y' + y = xy^3, y > 0$.

The logarithmic spiral. Suppose a curve $r = f(\theta)$ in polar coordinates cuts the radius at a constant angle, say ψ , as shown in the figure below.

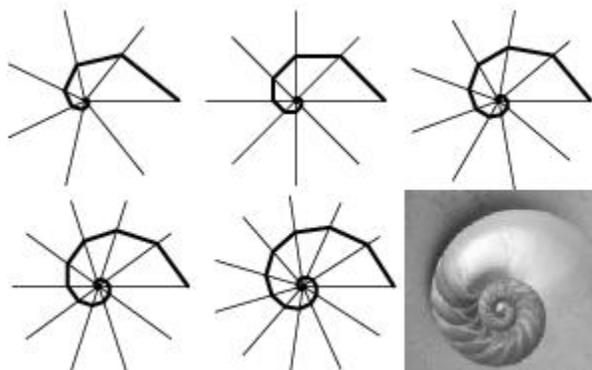


Figure 3.1. The logarithmic spiral.

The game is to find the equation of this curve. If $\psi = 0$ the curve is a ray extending from the origin to ∞ and it cannot be represented in the form $r = f(\theta)$. If $\psi = \pi/2$ or $-\pi/2$ the curve is a circle centered at the origin. If

$$(3.5) \quad -\pi/2 < \psi < \pi/2, \quad \psi \neq 0,$$

we write $\tan \psi = \frac{1}{k}$. The differential equation is

$$\tan \psi = \frac{r d\theta}{dr}.$$

(See Figure 3.2.)

Using $\tan \psi = \frac{1}{k}$ we obtain

$$(3.6) \quad \frac{r d\theta}{dr} = \frac{1}{k}, \quad \frac{dr}{d\theta} = kr,$$

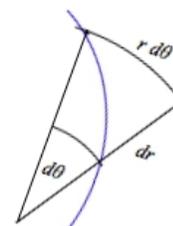


Figure 3.2.

and therefore $r = ce^{k\theta}$.

A curve of this kind is called a *logarithmic spiral*. The steps in the discussion above are reversible, and hence a curve $r = f(\theta)$ cuts the radius at a constant angle ψ satisfying (3.5) if, and only if, it is a logarithmic spiral. Since k is arbitrary, this gives a geometric interpretation of all processes obeying the exponential law of growth.

A logarithmic spiral looks like a snail shell (Figure 3.1), and this is not a coincidence. A snail shell has the characteristic that it grows only at one end, and the part of the shell already laid down does not change. The growth is specified by its rate in two perpendicular directions - radial

and transverse. Let us suppose that both rates are proportional to the size at time t . The precise meaning of "size" will not matter, and we use the weight W as a quantity that is easily measured. If arc length in the radial direction is s_1 and in the transverse direction is s_2 , then

$$(3.7) \quad \frac{ds_1}{dt} = k_1 W \quad \text{and} \quad \frac{ds_2}{dt} = k_2 W,$$

where k_1, k_2 are constants. The arc length in polar coordinates thus satisfies

$$ds^2 = dr^2 + (rd\theta)^2.$$

Setting $d\theta = 0$ we get $ds_1 = dr$ and setting $dr = 0$ we get $ds_2 = rd\theta$. Dividing the second equation in (3.7) by the first, we arrive at

$$\frac{rd\theta}{dr} = \frac{k_2}{k_1}.$$

This is the same equation as in (3.6) with $k = k_1/k_2$.