

MIT OpenCourseWare
<http://ocw.mit.edu>

18.034 Honors Differential Equations
Spring 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

LECTURE 0. TERMINOLOGY AND IMPLICIT SOLUTIONS

A *differential equation* (DE) is an equation between specified derivatives of an unknown function, its value, and known quantities and functions. Many physical laws are formulated as differential equations.

Ordinary differential equations are differential equations whose unknowns are functions of a single variable. They commonly arise in dynamical systems and electrical engineering. Partial differential equations are differential equations whose unknown depend two or more independent variables. In this course, we focus only on ordinary differential equations.

The *order* of a differential equation is the largest integer n , for which an n -th derivative occurs in the equation.

NOTATION. We typically use t or x for independent variables and y or u, v for unknowns, except for the plane systems of (parametric curves), for which we use t as the independent variable and x and y for unknowns. We use the prime to denote the differentiation. For instance, when t is the independent variable and y is the unknown, y' means $\frac{dy}{dt}$ and y'' means $\frac{d^2y}{dt^2}$.

In this note, we use t for the independent variable and y for the unknown.

The most general form of a differential equation of order n is

$$F(t, y, y', \dots, y^{(n)}) = 0.$$

A differential equation of order n is said of *normal form* if it takes the form

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}).$$

Differential equations are usually considered on an open interval $I = \{t : a < t < b\}$, where $-\infty \leq a < b \leq \infty$. A *solution* of a differential equation on I is a function $\phi(t)$ which, upon substitution $y = \phi(t)$, $y' = \phi'(t)$, \dots , satisfies the differential equation for every $t \in I$.

A differential equation is said *linear* if it is linear in the unknown and its derivatives. A linear differential equation of order n takes the form

$$p_0(t)y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = f(t),$$

where $p_j(t)$, $f(t)$ are continuous functions on an interval. It is said *homogeneous* if $f(t) = 0$. A differential equation is said *nonlinear* if it is not linear. Examples of nonlinear differential equations are $(y')^2 = t + y$ and $yy' = t$.

Many problems lead to two or more differential equations in two or more unknowns. In other words, they lead to a *system* of differential equations. For example,

$$x' = f(t, x, y), \quad y' = g(t, x, y).$$

Initial value problems. Differential equations commonly encountered in applications have infinitely many solutions. For example, $y' = f(t, y)$ has a family of solutions $y = \phi(t; c)$ depending on one parameter c and $y'' = f(t, y, y')$ has a family of solutions $y = \phi(t; c_1, c_2)$ depending on two parameters c_1 and c_2 . These parameters are like constants of integration, For example, we solve the differential equation $y'' = 0$ to obtain the family of solutions

$$y = c_1t + c_2,$$

depending on two constants of integration, c_1 and c_2 .

A simplest way to determine the parameters is to specify the value of y and its derivatives at a single point t_0 . For example, $y' = f(t, y)$ with $y(t_0) = y_0$, and $y'' = f(t, y, y')$ with $y(t_0) = y_0$, $y'(t_0) = y_1$. These equations are called *initial conditions* and the values y_j are called *initial values*. The reason for the term "initial value" is that in many problems t denotes the time and t_0 is the time at which the process starts.

An *Initial value problem* consists in finding the solution (or solutions) of the differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) \quad \text{for } t \geq t_0$$

satisfying

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \dots \quad y^{(n-1)}(t_0) = y_{n-1}.$$

Implicit solutions. Let us consider the differential equation

$$(0.1) \quad x + yy' = 0,$$

where $' = d/dx$. Since

$$\frac{d}{dx}(x^2 + y^2) = 2(x + yy')$$

the function $y = \phi(x)$ is a solution of (0.1) if and only if $x^2 + y^2 = c$, where c is a constant. In this sense the equation

$$(0.2) \quad x^2 + y^2 = c$$

defines solutions of (0.1) *implicitly* (with a function of x and y).

For $c < 0$ the locus of $x^2 + y^2 = c$ is empty, and it gives no solution. For $c = 0$ the locus consists of the single point $(x, y) = (0, 0)$. But, it does not give a solution since it does not give a differentiable function. For $c > 0$ the solution curve is the circle of radius \sqrt{c} centered at the origin.

Solving (0.2) for y , we obtain the (explicit) solution

$$y = \pm\sqrt{c - x^2},$$

which corresponds to the upper and the lower semicircles. These functions are defined for $-\sqrt{c} \leq x \leq \sqrt{c}$, but they are solutions of (0.1) only for $-\sqrt{c} < x < \sqrt{c}$. For, $y' = \mp \frac{x}{\sqrt{c - x^2}}$ becomes infinite at $x = \pm\sqrt{c}$.

The formulation (0.1) breaks down at $x = \pm\sqrt{c}$ and $y = 0$ since the points correspond to $dy/dx = \infty$. Nevertheless, the geometric interpretation remains meaningful. In the normal form of (0.1),

$$(0.3) \quad \frac{dy}{dx} = -\frac{x}{y},$$

y' means the slope of the solution curve and the right side gives the value of the slope at the point (x, y) . At $x = \pm\sqrt{c}$ and $y = 0$ the slope of the solution curve can be understood as being vertical.

To deal with this matter, note that a curve in the (x, y) -plane can be described not only by $y = \phi(x)$ but also by $x = \psi(y)$. The equation (0.3) implies that $y = \phi(x)$, where ϕ is differentiable and no solution curve of the equation can contain a point where $y = 0$, which would imply $dy/dx = \infty$. But, in the equation

$$\frac{dx}{dy} = -\frac{y}{x},$$

$y = 0$, which in turn implies $dx/dy = \psi'(y) = 0$, is permissible.

Nature has no recognizance of coordinate systems, which merely provide a framework for the mathematical description of an underlying reality. If a problem seems intractable when we insist on a solution $y = \phi(x)$, but easy when we allow $x = \psi(y)$, it could mean that we have made an inappropriate choice of independent and dependent variables in the formulation.