18.034 Honors Differential Equations
Spring 2009

Problem set 9, Solution keys

1. Birkhoff-Rota pp. 135-136, Theorem 1 and Example 3.

   Folia of Descartes are in figure 5.2.

2. (a) Suppose not. This means at least one of the inequalities $f > 0, f < 0, g > 0, g < 0$ holds at (x_1, y_1). Without loss of generality, let $f(x_1, y_1) = -2\alpha < 0$. (other cases are similar). By continuity, $f(x(t), y(t)) < -\alpha$ for $t > T$ for some large $T > 0$.

   Hence, $x'(t) \leq -\alpha$ for $t > T$, and $x(t) \leq -\alpha t + \beta$ for some β for $t > T$.

   Then $x(t) \to -\infty$ as $t \to \infty$, which contradicts that $x(t) \to x_1$.

   (b) Without loss of generality, let $x_0 = y_0 = 0$.

   Let $F(t) = f(xt, yt)$ in a disk $r < \delta$ for some $\delta > 0$. Hence $r = \sqrt{x^2 + y^2}$.

   By the Mean Value Theorem, $F(1) - F(0) = F'(\tau)$ for some $0 < \tau < 1$.

   And, $F'(\tau) = f_x(x\tau, y\tau)x + f_y(x\tau, y\tau)y$.

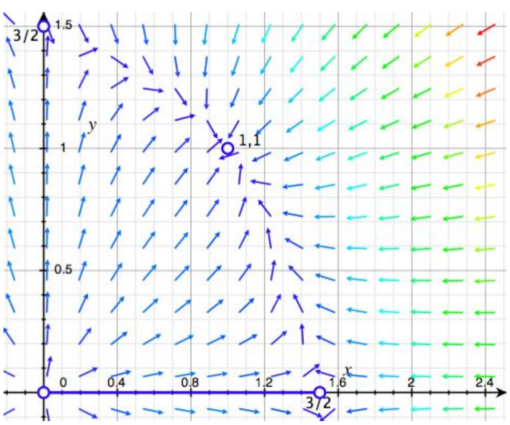
   Since f_x and f_y are continuous, $|f_x(x\tau, y\tau) - f_x(0,0)| < \epsilon(r)$, $|f_y(x\tau, y\tau) - f_y(0,0)| < \epsilon(r)$ and $\epsilon(r) \to 0$ as $r \to 0$.

   Therefore $a \to f_x(0,0)$, $\quad b \to f_y(0,0)$, as $r \to 0$. A similar argument applies to g.

3. $(0,0)$ unstable singular node

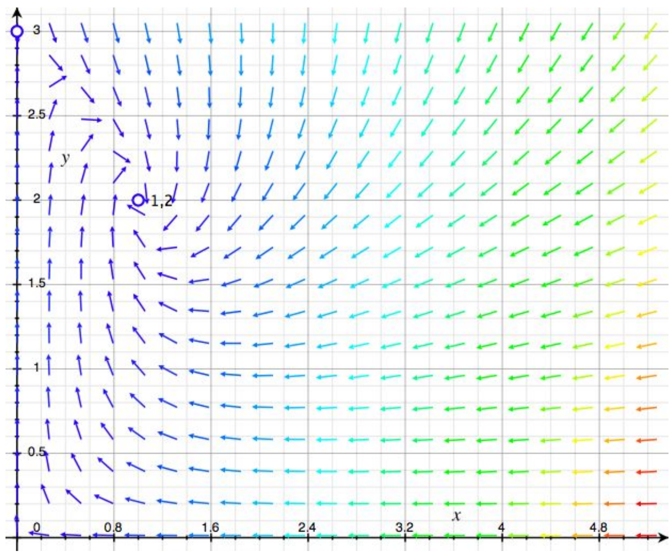
   $(\frac{3}{2}, 0), (0, \frac{3}{2})$ saddles.

   (1,1) stable nodes.



4. $(0,0)$, $\quad (0,3)$ saddles.

   (1,2) stable focus.



5. (a) Birkhoff-Rota pp. 153. Theorem 5.

   (b) Let $E(x) = x^2$ a Lyapunov function. $E(0) = 0$ and $E(x) > 0$ for $x \neq 0$.
   And $\dot{E}(x) = 2xf(x) < 0$ for $x \neq 0$ and x near 0 since $f'(0) < 0$.
   Therfore 0 is asymptotically stable.

6. (a) In addition to $x = 0, x = \frac{1}{n\pi},\ n = 1, 2, 3, ...$ are critical points. Let $o < |x(0)| \ll 1$ so that $\frac{1}{(n+1)\pi} < |x(0)| \leq \frac{1}{n\pi}$ for some n large.
   Then $\frac{1}{(n+1)\pi} < |x(t)| < \frac{1}{n\pi}$ for all t since a non-stationary solution cannot pass through a critical point.
   Therefore $|x(t)| < \frac{n+1}{n}|x(0)| \leq 2|x(0)|$, and 0 is stable.
   Take $x_n(0) = \frac{1}{n\pi}, n = 1, 2, 3...$. Then $x_n(t) = \frac{1}{n\pi}$ for all t. So ,0 is not asymptotically stable.

   (b) The linear system is $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
   The solutions are $x(t) = a + bt, y(t) = b$. Therefore $(0, 0)$ is unstable.
   For the nonlinear system, let $E(x, y) = x^4 + 2y^2$. Then, $E(0, 0) = 0, E(x, y) > 0$ for $(x, y) \neq (0, 0)$.
   $\dot{E}(x, y) = 4x^3(y - x^3) + 4y(-x^3) = -4x^6 \leq 0$. So, $(0, 0)$ is stable.