18.034 Honors Differential Equations
Spring 2009

Problem set 8, Solution keys

1. (a) Let $Au = \lambda u$. Then $u^*Au = u^*\lambda u = \lambda u u^*$.
   On the other hand, $u^*A = u^*A^* = (Au)^* = \bar{\lambda}u^*$. So, $u^*Au = \bar{\lambda}u^*u$. $\Rightarrow \lambda$ is real.

   (b) Let $Au = \lambda u,\ Av = \mu v, \qquad u \neq 0, v \neq 0, \lambda \neq \mu$
   Using the argument above, $v^*Au = \lambda v^*u, \quad u^*Av = \mu u^*v$.
   And $v^*Au = \bar{\mu}v^*u$. So $v^*v = 0$

2. $e^{At} = I + tA + \frac{t^2A^2}{2} + t^3R_1(t), \qquad e^{Bt} = I + tB + \frac{t^2B^2}{2} + t^3R_2(t)$

   So, $e^{At}e^{Bt} = I + t(A+B) + t^2(\frac{A^2+B^2}{2} + AB) + t^3R_3(t)$,

   $e^{Bt}e^{At} = I + t(A+B) + t^2(\frac{A^2+B^2}{2} + BA) + t^3R_4(t)$,

   Here $R_1(t), \cdots, R_4(t)$ are continuous matrix-valued functions.

   Therfore, $\lim_{t\to\infty} \dfrac{e^{At}e^{Bt} - e^{Bt}e^{At}}{t^2} = AB - BA \quad (= [A,B])$

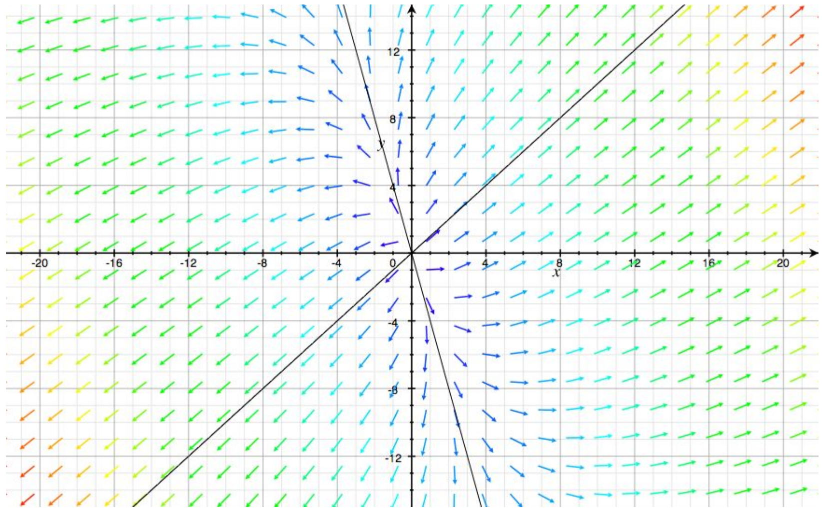
3. (a) $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$
   By Cayley-Hamilton, $(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) = 0$

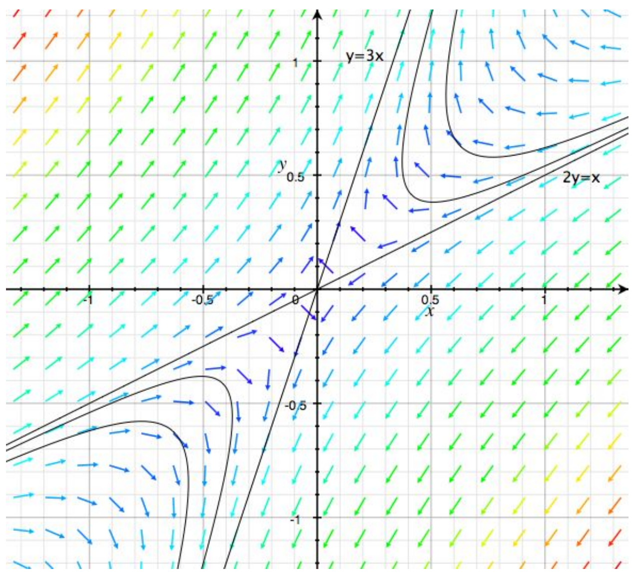
   (b) By Cayley-Hamilton, $A(A^2 - I) = 0$ So, $A^{2k+1} = A$ and $A^{2k} = A^2$.

$$
\begin{aligned}
e^{At} &= I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots \\
&= I + A(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots) + A^2(\frac{t^2}{2!} + \frac{t^4}{4!} + \cdots) \\
&= I + A^2 + \frac{1}{2}A(e^t - e^{-t}) + \frac{1}{2}A^2(e^t + e^{-t}).
\end{aligned}
$$

4. (a) $P(\lambda) = \lambda^2 - 9\lambda - 14$ ... unstable node.

   (b) $m = \frac{4+3m}{6+m}$, $m^2 + 3m - 4 = 0$ So, $m = 1$ or $m = -4$.

(c) 

5. (a) Saddle.
   (b) $m = \frac{1}{2}$ or 3.
   (c) 

6. Birkhoff-Rota pp.50-51.