18.034 Honors Differential Equations
Spring 2009

18.034 Solutions to Problemset 5

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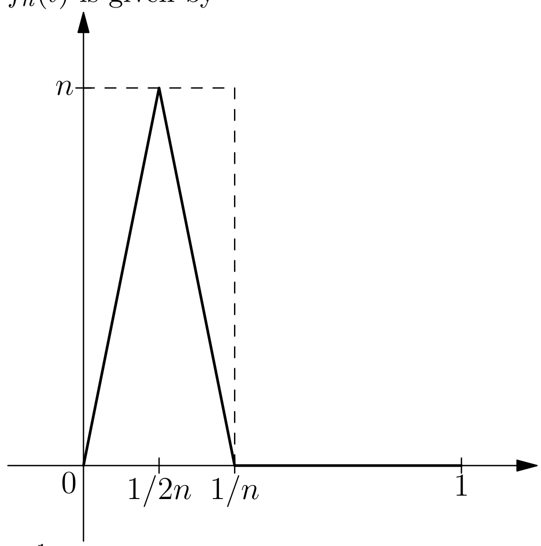
1. (a) Since $f_n \to f$ uniformly on $[a, b]$, for $\epsilon > 0$ given, there exists $N \in \mathbb{Z}_+$ such that $|f_n(t) - f(t)| < \epsilon/(b-a)$ for $t \in [a, b]$ whenever $n \geq N$. For $n \geq N$,

$$\left| \int_a^b f_n(t)\, dt - \int_a^b f(t)\, dt \right| \leq \int_a^b |f_n(t) - f(t)|\, dt < \epsilon$$

(b) $f_n(t)$ is given by



$\int_0^1 f_n(t)\, dt = \frac{1}{2}$ for all n,
$\int_0^1 f(t)\, dt = 0$.

2. In the course of the local existence theorem, $|x_k(t) - x_{k-1}(t)| \leq$

$ML^{k-1}\dfrac{|t-t_0|^k}{k!}$ for $k = 1, 2, \ldots$

$$\begin{aligned}
|x(t) - x_n(t)| &= \left|\sum_{k=n+1}^{\infty} (x_k(t) - x_{k-1}(t))\right| \\
&\le \sum_{k=n+1}^{\infty} |x_k(t) - x_{k-1}(t)| \\
&\le \tfrac{M}{L} \sum_{k=n+1}^{\infty} \tfrac{(LT)^k}{k!} \\
&= \tfrac{M}{L} \tfrac{(LT)^{n+1}}{(n+1)!} \sum_{k=0}^{\infty} \tfrac{(LT)^k}{k!} \\
&= ML^n \tfrac{T^{n+1}}{(n+1)!} e^{LT}
\end{aligned}$$

3. (a) ($\Rightarrow$) Let $F(t) = f(t, \phi(t))$ and solve $x'' = F(t)$ when $x(t_0) = x_0$, $x'(t_0) = x_1$.

   ($\Leftarrow$) Use $\dfrac{d}{dt}\displaystyle\int_{t_0}^{t} f(s,t)\,ds = f(t,t) + \int_{t_0}^{t} \frac{\partial f}{\partial t}(s,t)\,ds.$

   (b) Repeat the proof of the local existence theorem by showing

   $$\begin{aligned}
   (1)\quad |x_n(t) - x_0| &= |x_1|\,|t-t_0| + \left|\int_{t_0}^{t} (t-s) f(s, x_{n-1}(s)\,ds\right| \\
   &\le |x_1|\,|t-t_0| + M\int_{t_0}^{t} |t-s|\,ds \\
   &= |x_1|\,|t-t_0| + \tfrac{M}{2}|t-t_0|^2 \\
   &\le B|t-t_0|
   \end{aligned}$$

   $$\begin{aligned}
   (2)\quad |x_n(t) - x_{n-1}(t)| &\le \int_{t_0}^{t} (t-s)|f(s, x_{n-1}(s)) - f(s, x_{n-2}(s))|\,ds \\
   &\le L\int_{t_0}^{t} (t-s)|x_{n-1} - x_{n-2}|\,ds \\
   &\le ML^{n-1}\frac{|t-t_0|^{2n}}{(2n)!}
   \end{aligned}$$

   where L is the Lipshitz constant.

4. (a) $\mathcal{L}\left[\dfrac{1}{\sqrt{t}}\right] = \displaystyle\int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt = \int_0^\infty e^{-x^2}\frac{\sqrt{s}}{x}\frac{2x}{s}\,dx$, where $x^2 = st$.

   $= \dfrac{2}{\sqrt{s}}\displaystyle\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/s}$

   (b) $\mathcal{L}[\sqrt{t}] = \displaystyle\int_0^\infty e^{-st}\sqrt{t}\,dt = \sqrt{t}\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\frac{1}{2\sqrt{t}}\,dt$ by parts.

   $= 0 + \dfrac{1}{2s}\displaystyle\int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt = \sqrt{\pi}/2s^{3/2}$

5. (a) $\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2-st}\,dt$ is indefinite for every real value of s, no matter how large, since $t^2 - st > 0$ for $t > s$ and so

$$\int_0^\infty e^{t^2-st}\,dt > \int_s^\infty e^{t^2-st}\,dt > \int_s^\infty e^0\,dt = \infty$$

(b) $\mathcal{L}\left[\frac{1}{t^k}\right] = \int_0^\infty e^{-st}\frac{1}{t^k}\,dt$, $(s > 0)$. The trouble here is when $t = 0$. Near $t = 0$, $e^{-st} \approx 1$ and therefore $\int_0^\tau e^{-st}\frac{1}{t^k}\,dt \gtrsim \int_0^\tau \frac{dt}{t^k} = \begin{cases} \left.\frac{t^{1-k}}{1-k}\right|_0^\tau & k \neq 1 \\ \left.\log t\right|_0^\tau & k = 1 \end{cases}$. Therefore, $\mathcal{L}\left[\frac{1}{t^k}\right]$ exists for $k < 1$.

6. (a) $5\cos 2t - 3\sin 2t + 2$.

(b) $e^{-t/3}\cos\sqrt{2}t$.