18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #9

(modified on May 6, 2009)

Due by Friday, May 8, 2009, by NOON.

1. (a) If $V(x, y)$ is continuously differentiable, then each solution curve of the plane autonomous system

$$x' = \frac{\partial V}{\partial y}(x, y), \qquad y' = -\frac{\partial V}{\partial x}(x, y)$$

lies on some level curve $V(x, y) = constant$.

(b) Show that the solution curves of the autonomous system

$$x' = x(2y^3 - x^3), \qquad y' = -y(2x^3 - y^3)$$

are the curves $x^3 + y^3 - 3cxy = 0$, where c is an arbitrary constant. Sketch typical solution curves.

2. Consider the plane autonomous system

$$x' = f(x, y), \qquad y' = g(x, y),$$

where f, g are continuously differentiable.

(a) If $\lim_{t\to\infty}(x(t), y(t)) = (x_1, y_1)$, show that (x_1, y_1) is a critical point.

(b) Near a critical point (x_0, y_0) the system can be written as

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where a, b, c, d are functions of (x, y) satisfying

$$\lim_{(x,y)\to(x_0,y_0)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix}.$$

3. Find the critical points of the competitive system

$$x' = x(3 - 2x - y), \qquad y' = y(3 - x - 2y)$$

and determine the stability and behavior near the critical points.

4. Repeat Problem 3 for the predator-prey equation with self-limiting

$$x' = x(-1 - x + y), \qquad y' = y(3 - x - y).$$

5. (a) Show that $x' = f(x)$ has an asymptotically stable critical point 0 if and only if $0 < |x| < \delta$ implies $xf(x) < 0$ for some $\delta > 0$.

(b) Show that $x' = f(x)$ has an asymptotically stable critical point 0 if $f(0) = 0$ and $f'(0) < 0$.

6. (a) Let

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that the critical point 0 of $x' + f(x) = 0$ is neutrally stable, but not asymptotically stable.

(b) Show that the critical point $(0, 0)$ of the plane autonomous system

$$x' = y - x^3, \qquad y' = -x^3$$

is stable, although its linearization is unstable.