18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #8

(modified on April 27, 2009)

Due by Friday, May 1, 2009, by NOON.

1. An $n \times n$ complex matrix A is called *Hermitian* if $A = A^*$, where A^* is the conjugate transpose of A. That is, $a_{ij} = \overline{a}_{ji}$ for all $1 \leq i, j \leq n$. When A is real $A^* = A^T$ and the terms "Hermitian" and "symmetric" mean the same thing. If $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ are column vectors in $\mathbb{R}^n$, then $\vec{u}\vec{v}^* = u_1\bar{v}_1 + \cdots + u_n\bar{v}_n$ and $\|u\|^2 = uu^*$.

If $A = A^*$ show that all eienvalues of A are real. Furthermore, if $A = A^*$ then eigenvectors $\vec{u}$ and $\vec{v}$ corresponding to different eigenvalues λ and μ are orthogonal. That is, $\vec{u}\vec{v}^* = 0$.

2. If A and B are $n \times n$ matrices, compute

$$\lim_{t \to 0} \frac{e^{At}e^{Bt} - e^{Bt}e^{At}}{t^2}.$$

3. (a) Let A be a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Show that the nonzero columns of $(A - \lambda_2 I)(A - \lambda_3 I)$ are eigenvectors for λ_1.

(b) A 3×3 matrix A has characteristic polynomial $p(\lambda) = \lambda(\lambda^2 - 1)$. Find e^{At}.

4. (a) For $x' = 6x + y$, $y' = 4x + 3y$ show that the origin is an unstable node.

(b) If $y = mx$ is a trajectory, show that $m = 1$ or $m = -4$.

(c) Sketch the trajectories in the (x, y)-plane.

5. Repeat Problem 4 for $x' = -3x + 2y$, $y' = -3x + 4y$.

6. Consider the differential equation $u'' + p(t)u' + q(t)u = 0$, where $p(t), q(t)$ are continuous functions on some interval of t.

(a) Let

$$u(t) = r(t)\sin\theta(t), \qquad u'(t) = r(t)\cos\theta(t).$$

Show that

$$d\theta/dt = \cos^2\theta + p(t)\cos\theta\sin\theta + q(t)\sin^2\theta,$$
$$(1/r)dr/dt = -p(t)\cos^2\theta + (1 - q(t))\cos\theta\sin\theta.$$

(b) Using part (a) discuss that if $q(t) > p^2(t)/4$ then solutions are *oscillatory* and if $q(t) < 0$ then solutions are *nonoscillatory*.