18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #7

(modified on April 13, 2009)

Due by Friday, April 17, 2009, by NOON.

1. In many applications the input f and the output y are related by $L_2 y = L_1 f$, where L_1 and L_2 are linear differential operators.

(a) If P_j is the characteristic polynomial of L_j, $j = 1, 2$, show that

$$P_2(s)Y(s) = P_1(s)F(s) + P_0(s),$$

where $P_0(s)$is a polynomial depending on the initial conditions of both f and y.

(b) The null initial condition gives the transfer function $W(s) = P_1(s)/P_2(s)$. If $W(s) = \mathcal{L}w$, express y by the convolution theorem.

(c) We can have a function w as in part (b) only if the degree of P_1 is less than that of P_2. Why? What happens to the formula of part (b) if $P_1 = P_2$?

2. One method of determining the ratio e/m of charge to mass for an electron leads to the system

$$mx'' + Hey' = eE, \qquad my'' - Hex' = 0.$$

The initial values are $x(0) = x'(0) = y(0) = y'(0) = 0$ and m, e, H, E are positive constants.

(a) Solve by the Laplace transform and show that the path of a particle in the coordinates (x, y) are given by

$$x(t) = E(\omega H)^{-1}(1 - \cos \omega t), \qquad y(t) = E(\omega H)^{-1}(\omega t - \sin \omega t),$$

where $\omega = He/m$. This is a cycloid generated by a circle of radius $E(\omega H)^{-1}$.

(b) How can one determine the system-parameter e/m from knowledge of the path and the values E and H set by the experimenter?

3. Consider the system of differential equations $Y' = AY$, where

$$Y = \begin{pmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}.$$

(a) If $U(t)$ is a fundamental solution and C is a nonsingular matrix, show that UC is also a fundamental solution.

(b) If $V(t)$ is a fundamental solution, show that $U(t) = V(t)V(t_0)^{-1}$ satisfies $U(t_0) = I$ and is also a fundamental solution.

(c) Show that $|Y|' = D_1 + D_2$, where

$$D_1 = \begin{vmatrix} a_{11}y_{11} + a_{12}y_{21} & a_{11}y_{12} + a_{12}y_{22} \\ y_{21} & y_{22} \end{vmatrix} = a_{11}|Y|$$

and $D_2 = a_{22}|Y|$. Deduce the Liouville theorem.

4. (a) Show that if $(y_1(t), y_2(t))$ is a solution of

$$t^2 y_1' = -2ty_1 + 4y_2, \qquad ty_2' = -2ty_1 + 5y_2$$

then both y_1 and y_2 are solutions of the Euler equation

$$(t^2\phi' + 2t\phi)' = -8\phi + 5(t\phi' + 2\phi).$$

(b) Find a fundamental matrix for $\begin{pmatrix} -2t^{-1} & 4t^{-2} \\ -2 & 5t^{-1} \end{pmatrix}$

(c) For $A = \dfrac{1}{t^2}\begin{pmatrix} 2t & -2 \\ t^2+2t & -2 \end{pmatrix}$ obtain a fundamental matrix.

5. Solve the initial value problem

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

(a) by using the eigenvalues and (b) by using the Laplace transform.

(c) For which values of a and b, the solution of the initial value problem

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

has the behavior $\lim_{t\to\infty}(y_1(t), y_2(t))^T = (0,0)$?

6. For which vector (a,b,c), is the solution of the initial value problem

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

periodic in t?