18.034 Honors Differential Equations
Spring 2009

18.034 Problem Set #5

(modified on March 30, 2009)

Due by Friday, April 3, 2009, by NOON.

1. (a) Let $f_n(t)$, $n = 1, 2, \ldots$ be continuous functions on an interval $[a, b]$ and $\{f_n(t)\}$ converge uniformly to $f(t)$ on $[a, b]$. Show that

$$\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt.$$

(b) Construct $\{f_n(t)\}$ on $[0, 1]$ such that the above equality does not hold true.

2. For the initial value problem $x' = f(t, x)$ with $x(t_0) = x_0$, where f is continuous and Lipschitzian in the rectangle $|t - t_0| \le T$ and $|x - x_0| \le K$ with the Lipschitzian constant L, suppose the exact solution x and the Picard iterates x_n all exist over one and the same interval of t. Show that on such an interval

$$|x(t) - x_n(t)| \le ML^n \frac{T^{n+1}}{(n+1)!} e^{LT},$$

where $|f(t, x)| \le M$ in $|t - t_0| \le T$ and $|x - x_0| \le K$.

3. Let f be a real-valued continuous function in the rectangle $|t - t_0| \le T$ and $|x - x_0| \le K$. Consider the initial value problem

$$x'' = f(t, x), \qquad x(t_0) = x_0, \quad x'(t_0) = x_1. \tag{1}$$

(a) Show that ϕ is a solution of (1) if and only if ϕ is a solution of the integral equation

$$x(t) = x_0 + (t - t_0)x_1 + \int_{t_0}^t (t - s)f(s, x)ds. \tag{2}$$

(b) Let $\{x_n\}$ be a successive approximation for (2). That is, $x_0(t) = x_0$ and

$$x_n(t) = x_0 + (t - t_0)x_1 + \int_{t_0}^t (t - s)f(s, x_{n-1})ds \qquad n = 1, 2, \ldots.$$

If $f(t, x)$ is continuous and Lipshitzian with respect to x in the rectangle $|t - t_0| \le T$ and $|x - x_0| \le K$, show that $\{x_n\}$ converges on the interval $|t - t_0| \le \min(T, K/B)$ to the solution of (1), where $B = |x_1| + MT/2$ and $|f(t, x)| \le M$ in $|t - t_0| \le T$ and $|x - x_0| \le K$.

4. (a) Show that $\mathcal{L}(1/\sqrt{t})(s) = \sqrt{\pi/s}$ by using the well-known formula* $\displaystyle\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

(b) Use part (a) to show that $\mathcal{L}(\sqrt{t})(s) = \dfrac{\sqrt{\pi}}{2s^{3/2}}$ for $s > 0$.

5. (a) Show that $\mathcal{L}(e^{t^2})(s)$ does not exist for any interval of the form $s > a$.

(b) For what values of k, will $\mathcal{L}(1/t^k)$ exist?

6. Find the functions whose Laplace transforms are the following functions:

(a) $\dfrac{5s - 6}{s^2 + 4} + \dfrac{2}{s}$, (b) $\dfrac{9s + 3}{9s^2 + 6s + 19}$.

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*A mathematician is one to whom *that* is as obvious as that twice two makes four is to you. Liouville was a mathematician. – Lord Kelvin