18.034, Honors Differential Equations
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Lecture 23
4/2/04

0. Defined "integrable" and "of exponential type a". Spent the rest of the hour stating + proving properties of L. Used the notation $L[y(t)] = Y(s)$.

(1) Rigorously proved that if $y, y, \ldots, y^{(n)}$ are of exponential type so, then
$L[y^{(n)}] = s^n Y(s) - (y^{(n-1)}(0) + \ldots + s^{n-1}y(0)), s > s_0$.

(2) Computed directly that $L[1] = 1/s$, $s > 0$

(3) Used (1) to prove that for a polynomial of ∂'s n,
$L[p] = 1/s^{n+1} [p^{(n)}(0) + sp^{(n-1)}(0) + \ldots + s^{n-1}p(0)], s > 0$ (by using that $L[p^{(n+1)}] = 0$). In particular, $L[t^n] = n!/s^{n+1}$, $s > 0$.

(4) Used that sin(ωt) and cos(ωt) satisfy $y'' = \omega^2 y$, to deduce, $L[\sin(\omega t)] = \omega / s^2 + \omega^2$. Tested (1) by using that $\cos(\omega t) = -\omega \sin(\omega t)\ \cos(\omega t) = -\omega \sin(\omega t)$.

(5) For $a > 0, L[y(at)] = \frac{1}{a} Y\left(\frac{s}{a}\right)$ by chze-of-variable.

(6) Introduced notation for the unit step function
$s(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases}$ <u>Shift rule</u>: $L[s(t-t_0)y(t-t_0)] = e^{-st_0} Y(s)$ for $t_0>0$.

(7) $L[e^{at}y(t)] = Y(s-a)$.

(8) $L[R(t - t_0)] = e^{-st_0}$ for $t_0 \geq 0$.

(9) $L[t^n y(t)] = (-1)^n \frac{d^n}{ds^n} Y(s)$. Used this to double-check that $L[t^n] = n!/s^{n-1}$.

Used (8) + (1) to deduce $L\left[\frac{d}{ds} R(t - t_0)\right] = se^{-st_0}$ for $t_0 \geq 0$.

Discussed consistency of this with the derivation

$$\int_{-\infty}^{\infty} g(t).R'(t - t_0)dt = -\int_{-\infty}^{\infty} g'(t)R(t - t_0)dt = -g'(t_0)$$

for g a smooth, compactly supported function.