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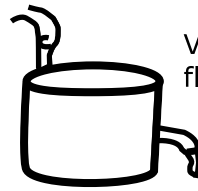
Lecture 2

1. Set-up model for a mixing problem
   Rate of mass of chemical in
   =(conc. in)×(rate of flow of liquid)= a.c(t)



V=volume, a=rate of flow of solution

Rate of mass out = (conc. out) ×(rate of flow) = $q \cdot \frac{y(t)}{V}$.

C(t)= concentration in

So $y' = a \cdot c(t) - \frac{a}{V} y$ ⟿ $\boxed{y' + \frac{a}{V} y = a.c(t)}$

y(t)= mass of chemical in tank at time t

where a, $V > 0$ are constants

2. Discussed method of integrating factors: $\boxed{y' + p(t)y = q(t)}$

   (a) Guess there exists $u(t)$ s.t. $u(t)y' + u(t)p(t)y$ equals $[u(t)y]$ '.
   (b) This leads to assorted separable equations, $u' = p(t)u$ which has a solution
   $U(t) = e^{\int p dt} = e^{\underline{p}(t)}$, which is evacuated.
   (c) Define $x = e^{\underline{p}(t)} y$, $y = e^{\underline{p}(t)} x$. Then $y' + p(t)y = q(t)$
   iff $x' = e^{\underline{p}(t)} q(t)$. Moreover, choosing $\underline{P}(0) = 0, y(0) = y_0$
   iff $x(0) = y_0$. So have existence/ uniqueness of original IVP
   iff existence/ uniqueness of IVP $x' = e^{\underline{p}(t)} q(t), x(0) = y_0$. But this follows from F.T. of calculus.
   (d) Conclusion: If $p(t), q(t)$ are defined and cts. on $(a,b) \subset \mathbb{R}$, then there exists a solution $y(t)$ of $y' + p(t)y = q(t)$ defined on all of (a,b), the solution is unique, and it has the form.

$$\boxed{y(t) = e^{-\underline{p}(t)} \int_0^t e^{\underline{p}(s)} q(s) ds + y_0 e^{-\underline{p}(t)},}$$

where $\underline{p}'(t) = p(t)$
$\underline{p}(0) = 0$

3. Used this method to solve the mixing problem:

$$y(t) = e^{\frac{-\alpha}{V} t} \int_0^t a e^{\frac{\alpha}{V} t} c(s) ds + y_0 e^{\frac{-\alpha}{V} t}$$

(a) If $c(t) = c$ is constant, get

$$y(t) = cV - (cV - y_0)\ e^{\frac{-\alpha}{V} t}, \text{ i.e.}$$

$$(cV - y(t)) = (cV - y_0)\ e^{\frac{-\alpha}{V} t}.$$

So equil. solution is $y = cV$ (which makes physical sense), and the "half-life", to come with $\frac{1}{2}$ of equilibrium from initial value is $\tau = \frac{V}{a} ln(2)$ (increasing V or decreasing a increases the half-life).

(b) Consider the case that $c(t) = A \cdot \sin(\omega t)$. Let to integral $\int_0^t aAe^{\frac{\alpha}{v}s} \sin(\omega s)\, ds$.

Set $\lambda = \frac{a}{v}$, get $\frac{aA}{\sqrt{\lambda^2 + \omega^2}} e^{\lambda t} \sin(\omega t - \phi)$, $\tan(\phi) = \frac{\omega}{\lambda}$. Didn't have time to really analyze the solution.

$$y(t) = \frac{aA}{\sqrt{\lambda^2 + \omega^2}} \sin(\omega t - \phi) + be^{-\lambda t} \text{ for some b}$$

4. Particular solution method. To find the general solution of $y' + p(t)y = q(t)$,
(i) Find general solution of undriven/ homo system $y_0' + p(t)y_0 = 0$.
(ii) Find a particular solution y_p of original equation.
(iii) General solution is $y_0 + y_p$.