18.034, Honors Differential Equations
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An application of the Dirac delta function 4/1/04

1. <u>A result from multivariable calculus</u>
Let (s,t) be coordinates on $\mathbb{R}^2$. Let $U \subset \mathbb{R}^2$ be an open region and let $a_1<b_1$, $a_2<b_2$ be real numbers such that the multi-interval $[a_1, b_1]\times[a_2, b_2]$ is contained in U. Let y (s, t) be a continuously differentiable function on U. Define the function y(t) on $[a_2, b_2]$ by,

$$y(t) = \int_{a_1}^{b_1} y(s,t)ds .$$

<u>Thm 1</u>: The function y(t) is continuously differentiable and $y'(t) = \int_{a_1}^{b_1} \frac{\partial y}{\partial t}(s,t)ds$.

The proof is found in most books on multivariable calculus, e.g., Theorem 5.1 of chapter 11 in Corwin + Szczarba, "Multivariable calculus".

Define the function z(s, t) on $[a_1, b_1]\times[a_2, b_2]$ by,

$$z(s, t)= \int_{a_1}^{s} y(u,t)du .$$

<u>Cor 2</u>: The function z(s, t) is continuously differentiable and $\frac{\partial z}{\partial s}(s,t) = y(s, t)$,

$$\frac{\partial z}{\partial t}(s,t) = \int_{a_1}^{s} \frac{\partial y}{\partial t}(u,t)du .$$

<u>Proof</u>: Clearly z(s, t) is continuous. Fix to in $[a_2, b_2]$. By the fundamental theorem of calculus, $z(s, t_0)$ is differentiable and $\frac{\partial z}{\partial s}(s,t_0) = y(s, t_0)$. So $\frac{\partial z}{\partial s}$ exists and is continuous. Fix s_0 in $[a_1, b_1]$. By Thm 1,

$\frac{\partial z}{\partial t}(s_0,t) = \int_{a_1}^{s_0} \frac{\partial y}{\partial t}(u,t)du$. So $\frac{\partial z}{\partial t}$ exists and is continuous. Therefore z(s, t) is continuously differentiable and the partial derivatives are $\frac{\partial z}{\partial s}(s, t) = y(s, t)$,

$$\frac{\partial z}{\partial t}(s, t)= \int_{a_1}^{s} \frac{\partial y}{\partial t}(u,t)du .$$

<u>Cor 3</u>: Suppose $a_1 = a_2 = a$ and $b_1 = b_2 = b$. The function z(t) = z (t, t) on [a, b] is continuously differentiable and

$$z'(t) = \int_{a}^{t} \frac{\partial y}{\partial t}(u,t)du + y(t, t).$$

Pf: By the Chain Rule, z(t, t) is continuously, diff. and

$$\frac{d}{dt} z(t) = \frac{\partial z}{\partial s}(t,t)\cdot 1 + \frac{\partial z}{\partial t}(t,t)\cdot 1$$

2.The Green's operator

Let $[t_0, t_1)$ be an interval (possibility $t_1=\infty$), let $a_0(t)$, $a_1(t)$,…., $a_n(t)$ be continuous functions on $[t_0, t_1)$, and let Ly be the differential operator,

$$Ly = y^{(n+1)} + a_n(t)y^0 + ... + a_1(t)y' + a_0(t)y \ .$$

For each real number s with $t_0 \le s < t_1$, denote by $\widetilde{K}_s(t)$ the unique (n+1)- times continuously differentiable function on $[t_0, t_1)$ that is the solution of the IVP

$$\begin{cases} L\widetilde{K}_s(t)=0, & \begin{array}{l} K_s(s)=0 \\ \vdots \\ \vdots \\ K_s^{(n-1)}(s)=0 \\ K_s^{(n)}(s)=1 \end{array} \end{cases}$$

Also, denote by $K_s(t)$ the function on $[t_0, t_1)$,

$$K_s(t) = \begin{cases} \widetilde{K}_s(t), & s \le t < t_1 \\ 0, & t_0 \le t < s \end{cases}$$

Denote by K (s, t) the function on $[t_0, t_1)\times[t_0, t_1)$ given by K (s, t) = $K_s(t)$. Similarly define $\widetilde{K}_s(t) = \widetilde{K}(s, t)$.

Prop 4: For each integer $0 \le m \le n$, the partial derivative $\frac{\partial^m \widetilde{K}}{\partial t^m}(s,t)$ exists and is continuously differentiable (here m=0 gives that $\widetilde{K}(s, t)$ is continuously differentiable).

Pf: This is easy to prove, but it will have to wait until we have discussed determinants and Cramer's rule.

Let f(t) be a continuously differentiable function on $[t_0, t_1)$ and define $y_d(t)$ by,

$$y_d(t)=\int_{t_0}^{t} \widetilde{K}(s,t)f(s)ds = \int_{t_0}^{t_1} K(s,t)f(s)ds \ .$$

Prop 5 : The function $y_d(t)$ is (n+1)- times continuously differentiable. Moreover, for m=1,….,n,

$$y_d^{(m)}(t) = \int_{t_0}^{t} \frac{\partial^m \widetilde{K}}{\partial t^m}(s,t)f(s)ds, \quad \text{and}$$

$$y_d^{(n+1)}(t) = \int_{t_0}^{t} \frac{\partial^{n+1} \widetilde{K}}{\partial t^{n+1}}(s,t)f(s)ds + f(t).$$

Pf : The proof is by induction. For the base case y(s, t)= $\widetilde{K}$ (s, t) f(s) satisfies the hypothesis of Cor 3. Therefore $y_d(t)$ is continuously differentiable and

$$y_d'(t) = \int_{t_0}^{t} \frac{\partial \widetilde{K}}{\partial t}(s,t) f(s) ds + \widetilde{K}(t,t) f(t).$$

By definition, K(t, t)=0. Thus $y_d'(t) = \int_{t_0}^{t} \frac{\partial \widetilde{K}}{\partial t}(s,t) f(s) ds$

Now let m be between 1 and n. By induction, assume that y_d is (m-1) – times continuously differentiable, and that $y_d^{(m)}(t)$ exists and $y_d^{(m)}(t) = \int_{t_0}^{t} \frac{\partial^m \widetilde{K}}{\partial t^m}(s,t) f(s) ds$.

By Prop 4, $\frac{\partial^m \widetilde{K}}{\partial t^m}(s,t) f(s)$ is continuously differentiable. Thus $y_d^{(m)}(t)$ is continuous.

And by Cor 3, $y_d^{(m)}(t)$ is continuously differentiable and

$$y_d^{(m+1)}(t) = \int_{t_0}^{t} \frac{\partial^{m+1} \widetilde{K}}{\partial t^{m+1}}(s,t) f(s) ds + \frac{\partial^m \widetilde{K}}{\partial t^m}(t,t) f(t).$$

If m ≤ n-1, then $\frac{\partial^m \widetilde{K}}{\partial t^m}(t,t) = 0$ by definition.

In this case $y_d^{(m+1)}(t) = \int_{t_0}^{t} \frac{\partial^{m+1} \widetilde{K}}{\partial t^{m+1}}(s,t) f(s) ds$.

In the case m=n, $\frac{\partial^n \widetilde{K}}{\partial t^n}(t,t) = 1$.

So $y_d^{(n+1)}(t) = \int_{t_0}^{t} \frac{\partial^{n+1} \widetilde{K}}{\partial t^{n+1}}(s,t) f(s) ds + f(t)$.

Remark 6:In fact Prop 5 is true if f(t) is only continuous. This can be proved on any bounded interval [t_0, t_1'] by taking a sequence of continuously differentiable functions $(f_k(t))$ that convergys uniformly to f(t), applying Prop 5 to each $f_k(t)$ and using standard results about integrals and derivatives of uniform limits.

Def'n 7: The function K(s, t) is called the Green's Kernel and the linear operator from $C^0[t_0, t_1)$ to $C^{n+1}[t_0, t_1)$ defined by

$$G[f](t) = y_d(t), \qquad y_d(t) = \int_{t_0}^{t} K(s,t) f(s) ds = \int_{t_0}^{t_1} K(s,t) f(s) ds$$

is called the Green's Operator (both "associated to L").

Thm 8: Let f(t) be a continuous function on $[t_0, t_1)$ and let $y_d(t)$= G[f] (t). Then $y_d(t)$ is the unique solution of the IVP

$$\begin{cases} L\, y_d(t)=f(t), & \begin{array}{l} y_d(t_0)=0 \\ \vdots \\ y_d^{(n)}(t_0)=0 \end{array} \end{cases}$$

Proof: By Prop 5, $y_d(t)$ is (n+1) – times continuously differentiable. From the equations for the first n derivatives of y_d, $y_d^{(m)}(t_0)=0$ for m=0,…., n. Finally, by the equations of the derivatives of y_d in Prop 5,

L y_d (t)= $a_0(t)y_d + a_1(t)y_d' + \ldots\ldots + a_n(t)y_d^{(n)} + y_d^{(n+1)}$, equals

$$a_0(t)\int_{t_0}^{t}\widetilde{K}(s,t)f(s)ds + \ldots\ldots + a_n(t)\int_{t_0}^{t}\frac{\partial^n\widetilde{K}}{\partial t^n}(s,t)f(s)ds + \int_{t_0}^{t}\frac{\partial^{n+1}\widetilde{K}}{\partial t^{n+1}}(s,t)f(s)ds + f(t)$$, equals

$$\int_{t_0}^{t}\left(a_0(t)\widetilde{K}(s,t) + a_1(t)\frac{\partial\widetilde{K}}{\partial t}(s,t) + \ldots\ldots + a_n(t)\frac{\partial^n\widetilde{K}}{\partial t^n}(s,t) + \frac{\partial^{n+1}\widetilde{K}}{\partial t^{n+1}}(s,t)\right)f(s)ds + f(t).$$

But for each s with $t_0 \le s \le t$,

$a_0(t)\widetilde{K}(s,t) + \ldots. + a_n(t)\dfrac{\partial^n\widetilde{K}}{\partial t^n}(s,t) + \dfrac{\partial^{n+1}\widetilde{K}}{\partial t^{n+1}}(s,t) = L\ \widetilde{K}_s(t)$ which equals 0 by definition.

So L y_d (t)= $\int_{t_0}^{t} 0\cdot f(s)ds + f(t) = 0 + f(t) = f(t)$.

Therefore y_d (t) is the solution of the IVP.

Remark 9: Even if f(t) isn't continuous, the Green's operator G[f] is often defined and is the best candidate for a solution of the IVP. For linear PDE's, a Green's operator is also defined and plays a central role in the study of linear PDE's.

3. Examples :(A) Let Ly = $y^{(n+1)}$. Then $\widetilde{K}_s(t)\dfrac{(t-s)^n}{n!}$.

So G[f] = $\displaystyle\int_{t_0}^{t}\frac{(t-s)^n}{n}f(s)ds$.

On the other hand, a solution of the IVP is clearly the integrated integral

$$\int_{t_0}^{t}\left(\cdots\int_{t_0}^{u_{n-1}}\left(\int_{t_0}^{u_n} f_{(u_{n+1})}du_{n+1}\right)du_n\cdots\right)du_1 .$$

Therefore, we have proved the identity

$$\boxed{\int_{t_0}^{t}\frac{(t-s)^n}{n!}f(s)ds = \int_{t_0}^{t}\left(\int_{t_0}^{u_1}\left(\cdots\int_{t_0}^{u_{n-1}}\left(\int_{t_0}^{u_n} f_{(u_{n+1})}du_{n+1}\right)du_n\cdots\right)du_2\right)du_1}$$

(B) Let n=1, Ly = y'' + a_1(t) y' + a_0(t). Let (y_1(t), y_2(t)) be a basic part of Ly=0. Then $\widetilde{K}_s(t)=\frac{y_1(s)}{W[y_1,y_2](s)}\cdot y_2(t)-\frac{y_2(s)}{W[y_1,y_2](s)}y_1(t)$ is a solution of L$\widetilde{K}_s(s)$=0. Also $\widetilde{K}_s(s)$=0 and $\widetilde{K}_s'(s)$=1. So the Green's Kernel is $\widetilde{K}_s(s,t)=\frac{y_1(s)y_2(t)-y_2(s)y_1(t)}{W[y_1,y_2](s)}$, and the Green's Operator is

$$\boxed{y_d(t)=\int_{t_0}^{t}\frac{[y_1(s)y_2(t)-y_1(t)y_2(s)]}{W[y_1,y_2](s)}F(s)ds.}$$

This is the same formula derived by "variation of parameters".

(C) Let Ly be a constant coefficient differential operator, i.e. Ly = p(D)y for some polynomial p(Z). Observe that for any fixed number s, p(D)(y(t-s))=(p(D)y) (t-s). So if y(t) is a solution of p(D)y=0, also y(t-s) is a solution.

Let $\widetilde{k}(t)=\widetilde{K}_0(t)$. Then for every s,
$\widetilde{k}(t-s)$ is a solution of the IVP

$$\begin{cases} & \widetilde{K}_s(s)=0 \\ L\widetilde{K}_s=0, & \vdots \\ & \widetilde{K}_s^{(n)}(s)=0 \\ & \widetilde{K}_s^{(n+1)}(s)=1 \end{cases}$$

So $\widetilde{K}_s(t)=\widetilde{k}(t-s)$. Similarly, define k (t)=K_0(t). Then K_s (t)=k (t-s).
Let $\widetilde{f}(t)$ be a continuous function on [t_0, t_1).

Define f(t)= $\begin{cases} \widetilde{f}(t), & t_0 \le t < t_1 \\ 0, & t < t_0 \text{ or } t \ge t_1 \end{cases}$

Then the solution of the IVP

$$\begin{cases} & \widetilde{y}(t_0)=0 \\ L\widetilde{y}=\widetilde{f}, & \vdots \\ & \widetilde{y}^{(n)}(t_0)=0 \end{cases}$$

on [t_0,t_1] is given by $\widetilde{y}=\int_{t_0}^{t}k(t-s)f(s)ds$

$=\int_{-\infty}^{\infty}k(t-s)f(s)ds=(k*f)(t)$.

So the Green's Operator is $\boxed{G[\widetilde{f}]=(k*f)}$.

This is another use of the convolution.

4. Derivation of Theorem 8. Except for a few minor complications, Theorem 8 is straightforward to prove. However, it is difficult to deduce from first principles. Why should there be a solution of the form $y_d(t) = \int_{t_0}^{t} K(s,t) f(s) ds$?

Why should K(s, t) satisfy the initial value problem that it does?

The most direct answer to both questions uses the Direct delta function R. The main observation is the following.

Observation. Let f(s, t) be a continuous function on $[s_0, s_1] \times [t_0, t_1)$. Let y(s, t) be a function on $[s_0, s_1] \times [t_0, t_1)$ such that y(s, t), $\frac{\partial y}{\partial t}(s,t), \ldots, \frac{\partial^{n+1} y}{\partial t^{n+1}}(s,t)$ are defined and continuous and s.t. for each s, L y(s,t) = f(s,t) , where t is the independent variable. Then y(t)= $\int_{s_0}^{s_1} y(s,t) ds$ is (n+1) – times continuously differentiable and

Ly(t)= $\int_{s_0}^{s_1} f(s,t) ds$.

This is not difficult to prove. Since the proof is not directly relevant, it is omitted. Now the continuous function is of the form $\int_{t_0}^{t_1} f(s,t) ds$, where f(s, t)=f(s)R (t-s).

Of course f(s, t) is a fictional function that is certainly not continuous. But the observation motivates us to hope we can make sense of the ODE Ly(s, t)=f(s)R (t-s) and that y(t)= $\int_{t_0}^{t_1} y(s,t) ds$ will be a solution of Ly(t)=f(t).

There is one straight forward reduction. Suppose we can make sense of Ly= R (t-s) and let K(s, t) be a solution. Then, by linearity of L, L(f(s)K(s,t))= f(s)L K(s,t) = f(s)R (t-s). So we can take y(s,t)= f(s)K(s,t) leading to the equation

y(t)= $\int_{t_0}^{t_1} k(s,t) f(s) ds$

We need to impose some conditions that are less easy to justify. We consider the IVP

$$\begin{cases} Ly = f(t) \\ y(t_0)=0 \\ \vdots \\ y^{(n)}(t_0)=0 \end{cases}.$$

We know that if f_a and f_b are 2 functions s.t. $f_a(t)= f_b(t)$ for $t_0 \le t \le t_2$, then $y_a(t)= y_b(t)$ for $t_0 \le t \le t_2$, i.e. the solution of the IVP on $[t_0, t_2]$ depends only on the value of f on $[t_0, t_2]$.

Therefore, for $t_0 \le t \le t_2$, $\int_{t_0}^{t_1} K(s,t)(f_b(s) - f_a(s)) ds$ =0, i.e. $\int_{t2}^{t_1} K(s,t)(f_b(s) - f_a(s)) ds = 0$ for $t \le t_2$. But the difference $f_b(s) - f_a(s)$ can be an arbitrary cts. function that is 0 on $[t_0,t_2]$. Taking $f_b(s) - f_a(s)$ limiting to $R(s-t_3)$ for $t_3>t_2$ gives the

$\int_{t_2}^{t_1} K(s,t)R(s-t_3)ds = 0$ for t ≥ t_2, i.e. $K(t_3, t)=0$ for $t_3 \geq t_2 \geq t$. But now t_3 and t are arbitrary as long as $t_3 > t$. This gives the strong restriction

K(s, t)=0 for s>t

So for t < s, K(s, t)=0. Since R(t-s)=0 for t > s, also for t > s K(s, t) is a solution of the homog. equation, L K(s, t)=0. If K(s, t) were n-times differentiable in t, it would follow that $\lim_{t\to\delta^+} \frac{\partial^t K}{\partial t^k}(s,t) = 0$ for k=0,…, n+1. This would imply that K(s, t) is <u>the</u> solution of the IVP LK(s, t)=0, K(s, s)=…..=$K^{(n+1)}$ (s, s)=0, namely K(s, t)=0.

This choice certainly does not lead to a solution of Ly=f(t)! Therefore, we should not expect K(s, t) to be (n+1)- times diff. It turns out the best we can hope for is that K(s, t) is (n-1)- times continuously differentiable and (n+1)- times piecewise continuously differentiable.

It may occur to the reader that we need not assume even this. The best response is that this is the assumption that leads to a true theorem. Another justification is nature: real-world systems that are modeled by linear differential operators do exhibit this behavior: If you kick a soccer ball, it does not instantaneously move 10 meters away. Rather the position of the ball is continuous and the velocity of the ball is piecewise continuous.

Since K,…., $K^{(n-1)}$ are continuous, for t > s this leads to the conditions, $\lim_{t\to\delta^+} \frac{\partial^k K}{\partial t^k}(s,t) = 0$ for k=0,…, n-1. The only missing piece of information is $\lim_{t\to\delta^+} \frac{\partial^n K}{\partial t^n}(s,t)$. Let us call this u(s).

To compute this, we reason heuristically as follows. The difference

$$K_s^{(n)}(\delta^+) - K_s^{(n)}(s^-) = \lim_{\varepsilon\to 0} \int_{s-\varepsilon}^{s+\varepsilon} K^{(n+1)}(t)dt$$

Of course $K_s^{(n+1)}(t)$ is only piecewise defined/cts, and it is not defined at t=s. However, our equation says

$$K_s^{(n+1)}(t) + a_n(t)K_s^{(n)}(t) + \ldots. + a_1(t)K'_s(t) + a_0(t)K_s(t) = \mathrm{R}$$

So we can solve for $K_s^{(n+1)}(t)$ and we get,

$$K_s^{(n+1)}(t) = \mathrm{R}\ (t-s) - (a_n(t)K_s^{(n)}(t) + \ldots + a_1(t)K'_s(t) + a_0(t)K_s$$

So $$\int_{s-\varepsilon}^{s+\varepsilon} K_s^{(n+1)}(t)dt = 1 - \left(\int_{s-\varepsilon}^{s+\varepsilon} a_n(t)K_s^{(n)}(t) + \ldots. + a_0(t)K_s(t)dt\right)$$

Now $K_s(t), \ldots., K_s^{(n-1)}(t)$ are all approximately 0 for ε small, and

$\lim_{\varepsilon \to 0} \int_{s-\varepsilon}^{s+\varepsilon} \left(a_0(t) K_s(t) + \ldots + a_n(t) K_s^n(t) dt\right)$ is 0. And $\int_{s-\varepsilon}^{s+\varepsilon} a_n(t) K_s^{(n)}(t) \approx \varepsilon \cdot a_n(s) U(s)$.

So $\lim_{\varepsilon \to 0} \int_{s-\varepsilon}^{s+\varepsilon} a_n(t) K_s^n(t) = 0$ as well.

In the limit, this gives

$$\lim_{\varepsilon \to 0} \int_{s-\varepsilon}^{s+\varepsilon} K^{(n+1)}(t) dt = \int_{s-\varepsilon}^{s+\varepsilon} R(t-s) dt = 1$$

So u(s) = 1.

This leads to the definition that $K_s(t)$ for $t \geq s$ is the unique solution of the IVP,

$$\begin{cases} L\, K_s(t) = 0 \\ K_s(s) = 0 \\ \quad \vdots \qquad \vdots \\ K_s^{(n-1)}(s) = 0 \\ K_s^{(n)}(s) = 1 \end{cases}$$

, and $K_s(t)=0$ for $t < s$. This is the definition that gives the formulation of Theorem 8.