18.034 SOLUTIONS TO PRACTICE EXAM 3, SPRING 2004

Problem 1 Let A be a 2×2 real matrix and consider the linear system of first order differential equations,

$$\mathbf{y}'(t) = A\mathbf{y}(t), \quad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

Let α be a real number, let β be a nonzero real number, and let M_1, M_2 be 2×2 matrices with real entries. Suppose that the general solution of the linear system is,

$$\mathbf{y}(t) = (k_1 M_1 + k_2 M_2) \begin{bmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{bmatrix},$$

where k_1, k_2 are arbitrary real numbers.

(a) Prove that M_1 and M_2 each satisfy the following equation,

$$AM_i = M_i D, \quad D = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

Solution: By assumption,

$$AM_i \begin{bmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{bmatrix} = M_i \frac{d}{dt} \begin{bmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{bmatrix}.$$

And,

$$\frac{d}{dt} \begin{bmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{bmatrix} = \begin{bmatrix} \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t) \\ \alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\sin(\beta t) \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{bmatrix}.$$

Therefore, for each real number t,

$$(AM_i - M_i D) \begin{bmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{bmatrix} = 0.$$

But for $t = 0$ and $t = \pi/(2\beta)$, the vectors give a basis for $\mathbb{R}^2$. Therefore $AM_i - M_i D = 0$.

(b) Consider the linear system of differential equations,

$$\mathbf{z}'(t) = A^2\mathbf{z}(t), \ \mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}.$$

Use (a) to show that for every pair of real numbers k_1, k_2, the following function is a solution of the linear system,

$$\mathbf{z}(t) = (k_1 M_1 + k_2 M_2) \begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t) \end{bmatrix}.$$

Solution: Because $AM_i = M_i D$, also

$$A^2 M_i = A(AM_i) = A(M_i D) = (AM_i)D = (M_i D)D = M_i D^2.$$

Now,

$$D^2 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha^2 - \beta^2 & -2\alpha\beta \\ 2\alpha\beta & \alpha^2 - \beta^2 \end{bmatrix}.$$

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And,

$$\frac{d}{dt}\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix} = \begin{bmatrix} \alpha^2-\beta^2 & -2\alpha\beta \\ 2\alpha\beta & \alpha^2-\beta^2\end{bmatrix}\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix}.$$

Thus,

$$\frac{d}{dt}M_i\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix} = M_iD^2\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix} = A^2M_i\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix}.$$

Therefore, for each pair of real numbers k_1, k_2,

$$\frac{d}{dt}(k_1M_1+k_2M_2)\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix} = A^2(k_1M_1+k_2M_2)\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix},$$

i.e.,

$$\mathbf{z}(t) = (k_1M_1+k_2M_2)\begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t)\end{bmatrix},$$

is a solution of $\mathbf{z}'(t) = A^2\mathbf{z}(t)$.

Problem 2 Consider the following inhomogeneous 2^{nd} order linear differential equation,

$$\begin{cases} y''-y=1, \\ y(0)=y_0, \\ y'(0)=v_0 \end{cases}$$

Denote by $Y(s)$ the Laplace transform,

$$Y(s) = \mathcal{L}[y(t)] = \int_0^\infty e^{-st}y(t)dt.$$

(a) Find an expression for $Y(s)$ as a sum of ratios of polynomials in s.

Solution: By rules of the Laplace transform, $\mathcal{L}[y'(t)] = sY(s)-y_0$ and $\mathcal{L}[y''(t)] = s^2Y(s)-sy_0-v_0$. Therefore,

$$(s^2Y(s)-sy_0-v_0)-Y(s) = \mathcal{L}[y''-y] = \mathcal{L}[1] = \frac{1}{s}.$$

Gathering terms and simplifying,

$$(s-1)(s+1)Y(s) = (s^2-1)Y(s) = v_0+sy_0+\frac{1}{s}.$$

Therefore,

$$Y(s) = \frac{s^2y_0+sv_0+1}{(s+1)s(s-1)}.$$

(b) Determine the partial fraction expansion of $Y(s)$.

Solution: Because each factor in the denominator is a linear factor with multiplicity 1, the Heaviside cover-up method determines all the coefficients,

$$\frac{s^2y_0+sv_0+1}{(s+1)s(s-1)} = \frac{y_0-v_0+1}{2}\frac{1}{s+1}+(-1)\frac{1}{s}+\frac{y_0+v_0+1}{2}\frac{1}{s-1}.$$

(c) Determine $y(t)$ by computing the inverse Laplace transform of $Y(s)$.

Solution: The inverse Laplace transform of $1/(s-a)$ is e^{at}. Therefore,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{y_0-v_0+1}{2}e^{-t}-1+\frac{y_0+v_0+1}{2}e^{t} = -1+(y_0+1)\cosh(t)+v_0\sinh(t).$$

Problem 3 The general *skew-symmetric* real 2×2 matrix is,

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix},$$

where b is a real number. Prove that the eigenvalues of A of the form $\lambda = \pm i\mu$ for some real number μ. Determine μ and find all values of b such that there is a single repeated eigenvalue.

Solution: The trace is $\text{Trace}(A) = 0$, and the determinant is $\det(A) = 0 - (-b^2) = b^2$. Therefore the characteristic polynomial is

$$p_A(\lambda) = \lambda^2 - \text{Trace}(A)\lambda + \det(A) = \lambda^2 + b^2.$$

Therefore the eigenvalues of A are $\pm ib$. There is a repeated eigenvalue iff $b = 0$.

There is a more involved proof that for every positive integer n, for every skew-symmetric real $n \times n$ matrix A, every eigenvalue of A is purely imaginary. The idea is that on $\mathbb{C}^n$ there is a *Hermitian inner product*, which assigns to each pair of vectors,

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},$$

the complex number,

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n.$$

Observe this has the properties,

$$\begin{aligned} \langle \mathbf{z}_1 + \mathbf{z}_2, \mathbf{w} \rangle &= \langle \mathbf{z}_1, \mathbf{w} \rangle + \langle \mathbf{z}_2, \mathbf{w} \rangle, \\ \langle \mathbf{z}, \mathbf{w}_1 + \mathbf{w}_2 \rangle &= \langle \mathbf{z}, \mathbf{w}_1 \rangle + \langle \mathbf{z}, \mathbf{w}_2 \rangle, \\ \langle \lambda \mathbf{z}, \mathbf{w} \rangle &= \lambda \langle \mathbf{z}, \mathbf{w} \rangle, \\ \langle \mathbf{z}, \lambda \mathbf{w} \rangle &= \overline{\lambda} \langle \mathbf{z}, \mathbf{w} \rangle, \\ \langle \mathbf{w}, \mathbf{z} \rangle &= \overline{\langle \mathbf{z}, \mathbf{w} \rangle}, \\ \langle \mathbf{z}, \mathbf{z} \rangle &\neq 0, \quad \text{if } \mathbf{z} \neq 0. \end{aligned}$$

Because A is a real skew-symmetric matrix, for every pair of vectors $\mathbf{z}, \mathbf{w}$ the following equation holds,

$$\langle A\mathbf{z}, \mathbf{w} \rangle = -\langle \mathbf{z}, A\mathbf{w} \rangle.$$

Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue and let $\mathbf{z}$ be a (nonzero) λ-eigenvalue. Then,

$$\lambda \langle \mathbf{z}, \mathbf{z} \rangle = \langle \lambda \mathbf{z}, \mathbf{z} \rangle = \langle A\mathbf{z}, \mathbf{z} \rangle = -\langle \mathbf{z}, A\mathbf{z} \rangle = -\langle \mathbf{z}, \lambda \mathbf{z} \rangle = -\overline{\lambda} \langle \mathbf{z}, \mathbf{z} \rangle.$$

Because $\mathbf{z}$ is nonzero, $\langle \mathbf{z}, \mathbf{z} \rangle$ is nonzero. Therefore $\lambda = -\overline{\lambda}$, which implies that λ is a pure imaginary number.

Problem 4 Let λ be a real number and let A be the following 3×3 matrix,

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Let a_1, a_2, a_3 be real numbers. Consider the following initial value problem,

$$\begin{cases} \mathbf{y}'(t) = A\mathbf{y}(t), \\ \mathbf{y}(0) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{cases}$$

Denote by $\mathbf{Y}(s)$ the Laplace transform of $\mathbf{y}(t)$, i.e.,

$$\mathbf{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix}, \quad Y_i(s) = \mathcal{L}[y_i(t)], \; i = 1, 2, 3.$$

(a) Express both $\mathcal{L}[\mathbf{y}'(t)]$ and $\mathcal{L}[A\mathbf{y}(t)]$ in terms of $\mathbf{Y}(s)$.

Solution: First of all,

$$\mathcal{L}[\mathbf{y}'(t)] = s\mathbf{Y}(s) - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Secondly,

$$\mathcal{L}[A\mathbf{y}(t)] = A\mathbf{Y}(s).$$

(b) Using part (a), find an equation that $\mathbf{Y}(s)$ satisfies, and iteratively solve the equation for $Y_3(s)$, $Y_2(s)$ and $Y_1(s)$, in that order.

Solution: By part (a),

$$s\mathbf{Y}(s) - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = A\mathbf{Y}(s).$$

Written out, this is equivalent to the system of 3 equations,

$$\begin{cases} (s-\lambda)Y_1(s) &= a_1 + Y_2(s) \\ (s-\lambda)Y_2(s) &= a_2 + Y_3(s) \\ (s-\lambda)Y_3(s) &= a_3 \end{cases}$$

Solving this iteratively,

$$Y_3(s) = \frac{a_3}{s-\lambda},$$

$$Y_2(s) = \frac{a_2}{s-\lambda} + \frac{1}{s-\lambda}Y_3(s) = \frac{a_2}{s-\lambda} + \frac{a_3}{(s-\lambda)^2},$$

and,

$$Y_1(s) = \frac{a_1}{s-\lambda} + \frac{1}{s-\lambda}Y_2(s) = \frac{a_1}{s-\lambda} + \frac{a_2}{(s-\lambda)^2} + \frac{a_3}{(s-\lambda)^3}.$$

(c) Determine $\mathbf{y}(t)$ by applying the inverse Laplace transform to $Y_1(s)$, $Y_2(s)$ and $Y_3(s)$.

Solution: The relevant inverse Laplace transforms are,

$$\begin{aligned} \mathcal{L}^{-1}[1/(s-\lambda)] &= e^{\lambda t}, \\ \mathcal{L}^{-1}[1/(s-\lambda)^2] &= te^{\lambda t}, \\ \mathcal{L}^{-1}[1/(s-\lambda)^3] &= \tfrac{1}{2}t^2 e^{\lambda t} \end{aligned}$$

Therefore,

$$\begin{cases} y_1(t) &= a_1 e^{\lambda t} + a_2 t e^{\lambda t} + a_3 \frac{1}{2} t^2 e^{\lambda t}, \\ y_2(t) &= a_2 e^{\lambda t} + a_3 t e^{\lambda t}, \\ y_3(t) &= a_3 e^{\lambda t} \end{cases}$$

In matrix form, this is,

$$\mathbf{y}(t) = a_1 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 e^{\lambda t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + a_3 e^{\lambda t} \begin{bmatrix} \frac{t^2}{2} \\ t \\ 1 \end{bmatrix}.$$

Problem 5 For each of the following matrices A, compute the following,

(i) Trace(A),
(ii) $\det(A)$,
(iii) the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$,
(iv) the eigenvalues of A (both real and complex), and
(v) for each eigenvalue λ a basis for the space of λ-eigenvectors.

(a) The 2×2 matrix with real entries,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Hint: See Problem 3.

Solution: In Problem 3, we computed Trace$(A) = 0$, $\det(A) = 1$, $p_A(\lambda) = \lambda^2 + 1$, and the eigenvalues are $\lambda_\pm = \pm i$. For the eigenvalue $\lambda_+ = i$, denote an eigenvector by,

$$\mathbf{v}_+ = \begin{bmatrix} v_{+,1} \\ v_{+,2} \end{bmatrix}.$$

Then $-v_{+,2} = iv_{+,1}$, e.g., $v_{+,1} = 1, v_{+,2} = -i$. Therefore an eigenvector for $\lambda_+ = i$ is,

$$\mathbf{v}_+ = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Similarly, an eigenvector for $\lambda_- = -i$ is,

$$\mathbf{v}_- = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

(b) The 3×3 matrix with real entries,

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution: Because this is an upper triangular matrix, clearly Trace$(A) = 3 + 5 + 3 = 11$, $\det(A) = 3 \times 5 \times 3 = 45$, and $p_A(\lambda) = (\lambda - 3)(\lambda - 5)(\lambda - 3) = \lambda^3 - 11\lambda + 39\lambda - 45$. The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 3$ (the eigenvalue 3 has multiplicity 2).

For the eigenvalue $\lambda_1 = 5$, the eigenvectors are the nonzero nullvectors of the matrix,

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Either by using row operations to put this matrix in row echelon form, or by inspection, a basis for the nullspace is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = 3$, the eigenvectors are the nonzero nullvectors of the matrix,

$$A - 3I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case, a basis for the nullspace is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$