18.034 SOLUTIONS TO PRACTICE EXAM 2, SPRING 2004

Problem 1 Let r be a positive real number. Consider the 2^{nd} order, linear differential equation,

$$y'' - \left(r + \frac{3}{t}\right) y' + \left(\frac{2r}{t} + \frac{3}{t^2}\right) y = 0,$$

where $y(t)$ is a function on $(0, \infty)$. One solution of this equation is $y_1(t) = te^{rt}$. Use Wronskian reduction of order to find a second solution $y_2(t)$.

Solution For the Wronskian $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$, differentiating gives,

$$W' = -a(t)W = \left(r + \frac{3}{t}\right) W.$$

This is a separable equation whose solution is,

$$\ln(W) = rt + 3\ln(t) + C,$$

in other words,

$$W(t) = At^3 e^{rt}.$$

Without loss of generality, take $A = 1$.

By definition $v = y_2(t)$ is a solution of the following 1^{st} order ODE,

$$te^{rt}v' - (rt + 1)e^{rt}v = t^3 e^{rt}.$$

Putting this in normal form,

$$v' + (-r - \frac{1}{t})v = t^2.$$

An integrating factor for this equation is,

$$\begin{aligned} u(t) &= \exp\left[\textstyle\int_{t_0}^{t} (-r - \frac{1}{s})ds\right] \\ &= \exp\left[-rt - \ln(t) + B\right] \\ &= Ct^{-1}e^{-rt}, \end{aligned}$$

where C is a constant. Set $C = 1$.

The integrating factor reduces the ODE to,

$$\left[t^{-1}e^{-rt}v\right]' = te^{-rt}.$$

Integrating by parts, the antiderivative of te^{-rt} is,

$$\int te^{-rt}dt = -\frac{1}{r^2}(rt + 1)e^{-rt} + E.$$

Hence,

$$t^{-1}e^{-rt}v = -\frac{1}{r^2}(rt + 1)e^{-rt} + E.$$

One solution is,

$$v(t) = -\frac{1}{r^2}t(rt + 1).$$

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Of course any multiple of this solution also leads to a basic solution set. Therefore a basic solution set of the ODE,

$$y'' - \left(r + \frac{3}{t}\right) y' + \left(\frac{2r}{t} + \frac{3}{t^2}\right) y = 0,$$

is the pair,

$$y_1(t) = te^{rt}, \quad y_2(t) = t(rt+1).$$

Problem 2 An undamped harmonic oscillator satisfies the ODE,

$$y'' + \omega^2 y = 0.$$

Let $y(t)$ be a solution of this ODE for $t < \tau$. At some time $\tau > 0$, the oscillator is given an *impulse* of size $v > 0$. In other words, if

$$\begin{cases} \lim_{t \to \tau^-} y(t) & = & y_0, \\ \lim_{t \to \tau^-} y'(t) & = & v_0 \end{cases}$$

then for $t > \tau$, $y(t)$ is a solution of the IVP,

$$\begin{cases} y'' + \omega^2 y = 0, \\ y(\tau) = y_0, \\ y'(\tau) = v_0 + v \end{cases}$$

(a) Write $y(t)$ in normal form $A\cos(\omega t - \phi)$ for $t < \tau$, and in normal form $y(t) = B\cos(\omega t - \psi)$ for $t > \tau$. Find an equation expressing B^2 in terms of A^2, v_0 and v.

Solution For a function $z(t)$ in the form $C\cos(\omega t - \theta)$, the derivative is $z'(t) = -\omega C \sin(\omega t - \theta)$. In particular,

$$(\omega z)^2 + (z')^2 = \omega^2 C^2 \cos^2(\omega t - \theta) + \omega^2 C^2 \sin^2(\omega t - \theta) = \omega^2 C^2.$$

In particular,

$$\begin{gathered} \omega^2 B^2 = (\omega y(\tau))^2 + (y'(\tau))^2 \\ = (\omega y_0)^2 + (v_0 + v)^2 = (\omega y_0)^2 + v_0^2 + 2v_0 v + v^2 \\ = \omega^2 A^2 + 2v_0 v + v^2. \end{gathered}$$

This gives the formula,

$$B^2 = A^2 + 2\frac{1}{\omega^2} v_0 v + \frac{1}{\omega^2} v^2.$$

(b) If the goal of the impulse is to maximize the amplitude B, at what moment τ in the cycle of the oscillator should the impulse be applied? If the goal is minimize the amplitude B, at what moment τ should the impulse be applied?

Solution Maximizing B is the same as maximizing B^2. In the equation above, A^2, ω and v are the same for all values of τ. The only quantity that varies is v_0. To maximize B^2, the impulse should be applied when v_0 is as large as possible, at the moment when $y_0 = 0$ and $y'(t) > 0$. In other words, when

$$\omega\tau - \phi = (2n - 1/2)\pi, \quad \tau = \frac{1}{\omega}(\phi + (2n - 1/2)\pi).$$

Similarly, to minimize B, the impulse should be applied when v_0 is as negative as possible, at the moment when $y_0 = 0$ and $y'(t) < 0$. In other words, when

$$\omega\tau - \phi = (2n + 1/2)\pi, \quad \tau = \frac{1}{\omega}(\phi + (2n + 1/2)\pi).$$

Problem 3 Consider the following constant coefficient linear ODE,

$$y''' + y = 0.$$

(a) Find the characteristic polynomial and find all real and complex roots.

Solution The characteristic polynomial is,

$$p(z) = z^3 + 1.$$

One evident root is $z = -1$. Factoring this out gives,

$$z^3 + 1 = (z+1)(z^2 - z + 1).$$

By the quadratic formula, the two roots of $z^2 - z + 1$ are the complex conjugates,

$$\lambda_\pm = 1/2 \pm i\sqrt{3}/2.$$

(b) Find the general *real-valued* solution of the ODE.

Solution Associated to the root -1 is the real-valued solution e^{-t}. Associated to the complex conjugates $\lambda_\pm$ are the two real solutions,

$$e^{t/2}\cos(\sqrt{3}t/2), \quad e^{t/2}\sin(\sqrt{3}t/2).$$

Therefore the general real-valued solution is,

$$y_g(t) = C_1 e^{-t} + C_2 e^{t/2}\cos(\sqrt{3}t/2) + C_3 e^{t/2}\sin(\sqrt{3}t/2).$$

(c) Find a particular solution of the driven ODE,

$$y''' + y = \cos(\sqrt{3}t/2).$$

Solution A particular solution is the real part of the complex-valued solution of the driven complex ODE,

$$\widetilde{y}''' + \widetilde{y} = e^{i\sqrt{3}t/2}.$$

Because $\frac{i\sqrt{3}}{2}$ is not a root of the characteristic polynomial, we guess the solution is of the form,

$$\widetilde{y} = Ae^{i\sqrt{3}t/2}.$$

Substituting this into the ODE gives,

$$(i\sqrt{3}/2)^3 Ae^{i\sqrt{3}t/2} + Ae^{i\sqrt{3}t/2} = e^{i\sqrt{3}t/2}.$$

Simplifying gives,

$$A(1 - 3\sqrt{3}i/8) = 1,$$

i.e.,

$$\frac{1}{8}A(8 - 3\sqrt{3}i) = 1.$$

Multiplying both sides by the complex conjugate $8 + 3\sqrt{3}i$ gives,

$$\frac{1}{8}A(64 - 27) = (8 + 3\sqrt{3}i),$$

i.e.

$$A = \frac{8}{37}(8 + 3\sqrt{3}i).$$

So the real part of $\widetilde{y}(t)$ is,

$$y_d(t) = \frac{8}{37}(8\cos(\sqrt{3}t/2) - 3\sqrt{3}\sin(\sqrt{3}t/2)).$$

Problem 4 The linear ODE,

$$y'' + (t - 3/t)y' - 2y = 0,$$

has a basic solution pair $y_1(t) = e^{-t^2/2}$, $y_2(t) = t^2 - 2$.

(a) Find the Wronskian $W[y_1, y_2](t)$.

Solution Computing the derivatives,

$$\begin{array}{rclrcl} y_1(t) &=& e^{-t^2/2}, & y_2(t) &=& t^2-2, \\ y_1'(t) &=& -te^{-t^2/2}, & y_2'(t) &=& 2t. \end{array}$$

So the Wronskian is,

$$2te^{-t^2/2} - (-t)(t^2-2)e^{-t^2/2} = t^3 e^{-t^2/2}.$$

(b) Use variation of parameters to find a particular solution of the driven ODE,

$$y'' + (t - 3/t)y' - 2y = t^4.$$

Solution By variation of parameters, a particular solution of $Ly = f(t)$ is,

$$y_d(t) = \int_{t_0}^{t} K(t,s) f(s) ds,$$

where,

$$K(t,s) = (y_1(s)y_2(t) - y_1(t)y_2(s))/W[y_1,y_2](s).$$

By (a), $W(s) = s^3 e^{-s^2/2}$. Therefore,

$$K(t,s) = (e^{-s^2/2}(t^2-2) - e^{-t^2/2}(s^2-2))/(s^3 e^{-s^2/2}).$$

Simplifying, this is,

$$K(t,s) = \frac{1}{s^3}(t^2-2) - e^{-t^2/2}\left(\frac{s^2-2}{s^3}\right)e^{s^2/2}.$$

Multiplying by s^4 yields,

$$K(t,s)s^4 = (t^2-2)s - e^{-t^2/2}(s^3-2s)e^{s^2/2}.$$

The antiderivative of the first term is,

$$\int_{t_0}^{t} (t^2-2) s ds = \frac{1}{2}(t^2 - t_0^2)(t^2-2).$$

To antidifferentiate the second term, substitute $u = s^2/2$, $du = sds$ to get,

$$\int_{t_0^2/2}^{t^2/2} -e^{-t^2/2}(u-2)e^u du.$$

Integrating by parts, this is,

$$\begin{array}{c} \int_{t_0^2/2}^{t^2/2} -e^{-t^2/2}(u-2)e^u du = \\ -e^{-t^2/2}\left((u-3)e^u\right|_{t_0^2/2}^{t^2/2} = \\ -e^{-t^2/2}\left(\frac{1}{2}(t^2-6)e^{t^2/2} - \frac{1}{2}(t_0^2-6)e^{t_0^2/2}\right) = \\ -\frac{1}{2}(t^2-6) + \frac{1}{2}(t_0^2-6)e^{t_0^2/2}e^{-t^2/2}. \end{array}$$

Putting the pieces together and plugging in $t_0 = 0$ gives,

$$y_d(t) = \frac{1}{2}(t^4 - 3t^2 + 6) - 3e^{-t^2/2}.$$

It is straightforward to check this is a solution.

Problem 5 Recall that $\mathrm{PC}_{\mathbb{R}}(0,1]$ is the set of all piecewise continuous real-valued functions on the interval $(0,1]$. The inner product on this set is,

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Define $f_0(t) = 1$. For each integer $n \geq 1$, define $f_n(t)$ to be the piecewise continuous function whose value on $(0, \frac{1}{2^n}]$ is -1, whose value on $(\frac{1}{2^n}, \frac{2}{2^n}]$ is $+1$, whose value on $(\frac{2}{2^n}, \frac{3}{2^n}]$ is -1, whose value on $(\frac{3}{2^n}, \frac{4}{2^n}]$ is $+1$, etc. In other words,

$$f_n(t) = \begin{cases} -1, & \frac{2k-2}{2^n} < t \leq \frac{2k-2}{2^n} \quad \text{for } k = 1, \dots, 2^{n-1}, \\ \\ +1, & \frac{2k-1}{2^n} < t \leq \frac{2k}{2^n} \quad \text{for } k = 1, \dots, 2^{n-1}. \end{cases}$$

(a) Compute the integrals $\langle f_m, f_n \rangle$ and use this to prove that $(f_0, f_1, \dots)$ is an orthonormal sequence. (**Hint**: If $n > m$, consider the integral of f_n over one of the subintervals $(\frac{a}{2^m}, \frac{a+1}{2^m}]$. What fraction of the time is f_n positive and what fraction of the time is it negative?)

Solution First of all, for every n, $(f_n(t))^2$ is the constant function 1. Therefore $\langle f_n, f_n \rangle = 1$. Suppose that $n > m$. Then the integral $\langle f_n, f_m \rangle$ is the sum over all integers $a = 0, \dots, 2^m - 1$ of the integral,

$$\int_{a/2^m}^{(a+1)/2^m} \pm f_n(t) dt.$$

Of course the interval $(\frac{a}{2^m}, \frac{a+1}{2^m}]$ is a union of 2^{n-m} intervals $(\frac{b}{2^n}, \frac{b+1}{2^n}]$. On half of these intervals, $f_n(t)$ has the constant value -1. On the other half, $f_n(t)$ has the constant value $+1$. Therefore the net integral of $f_n(t)$ over $(\frac{a}{2^m}, \frac{a+1}{2^m}]$ is 0. Since this holds for each a,

$$\langle f_n, f_m \rangle = 0.$$

Therefore the sequence $(f_0, f_1, \dots)$ is an orthonormal sequence.

(b) Compute the *generalized Fourier coefficient*,

$$\langle t, f_n(t) \rangle = \int_0^1 t f_n(t) dt.$$

Prove it equals $\frac{1}{2^{n+1}}$. This gives the generalized Fourier series,

$$t = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f_n(t).$$

Solution Of course for $n = 0$, $\langle t, f_0(t) \rangle$ is just the integral of t, which is $\frac{1}{2}$. Suppose that $n > 0$. By definition,

$$\langle t, f_n(t) \rangle = \sum_{k=1}^{2^{n-1}} \left(\int_{(2k-2)/2^n}^{(2k-1)/2^n} t(-1) dt + \int_{(2k-1)/2^n}^{2k/2^n} t(+1) dt \right).$$

Integrating, this is,

$$\sum_{k=1}^{2^{n-1}} \left(-\left(t^2/2\right|_{(2k-2)/2^n}^{(2k-1)/2^n} + \left(t^2/2\right|_{(2k-1)/2^n}^{2k/2^n} \right).$$

The term in parentheses simplifies to,

$$\begin{gathered} -\frac{1}{2}\left((2k-1)^2/2^{2n} - (2k-2)^2/2^{2n}\right) + \frac{1}{2}\left((2k)^2/2^{2n} - (2k-1)^2/2^{2n}\right) = \\ \frac{1}{2^{2n+1}}\left((2k)^2 - 2(2k-1)^2 + (2k-2)^2\right) = \\ \frac{1}{2^{2n+1}}\left(4k^2 - 2(4k^2 - 4k + 1) + (4k^2 - 8k + 4)\right) = \\ \frac{1}{2^{2n+1}}\left(4k^2 - 8k^2 + 8k - 2 + 4k^2 - 8k + 4\right) = \\ \frac{1}{2^{2n}}. \end{gathered}$$

Summing over all k gives $2^{n-1} \times (1/2^{2n}) = 1/2^{n+1}$. Therefore the generalized Fourier coefficient is,

$$\langle t, f_n(t) \rangle = \frac{1}{2^{n+1}}.$$

This gives the generalized Fourier series,

$$t = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f_n(t).$$

(c) Rewrite the series above as,

$$t = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1 + f_n(t)}{2}.$$

What is the relationship of this equation to the binary expansion of the real number t?

Solution We can rewrite the equation because,

$$\frac{1}{2} f_0 = \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}.$$

Now $1 + f_n(t)$ equals 0 iff the n^{th} digit in the binary expansion of t equals 0. And $1 + f_n(t)$ equals 2 iff the n^{th} digit in the binary expansion of t equals 1. Therefore $(1 + f_n(t))/2$ is precisely the n^{th} digit in the binary expansion of t. Therefore the formula above precisely says that t is equal to the series arising from the binary expansion of t.