18.034 PRACTICE EXAM 3, SPRING 2004

Problem 1 Let A be a 2×2 real matrix and consider the linear system of first order differential equations,

$$\mathbf{y}'(t) = A\mathbf{y}(t), \quad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

Let α, β be fixed real numbers, and let M_1, M_2 be fixed 2×2 matrices with real entries. Suppose that the general solution of the linear system is,

$$\mathbf{y}(t) = (k_1 M_1 + k_2 M_2) \begin{bmatrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{bmatrix},$$

where k_1, k_2 are arbitrary real numbers.

(a) Prove that M_1 and M_2 each satisfy the following equation,

$$AM_i = M_i D, \quad D = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

(b) Consider the linear system of differential equations,

$$\mathbf{z}'(t) = A^2\mathbf{z}(t), \ \mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}.$$

Use (a) to show that for every pair of real numbers k_1, k_2, the following function is a solution of the linear system,

$$\mathbf{z}(t) = (k_1 M_1 + k_2 M_2) \begin{bmatrix} e^{(\alpha^2-\beta^2)t}\cos(2\alpha\beta t) \\ e^{(\alpha^2-\beta^2)t}\sin(2\alpha\beta t) \end{bmatrix}.$$

Problem 2 Consider the following inhomogeneous 2^{nd} order linear differential equation,

$$\begin{cases} y'' - y = 1, \\ y(0) = y_0, \\ y'(0) = v_0 \end{cases}$$

Denote by $Y(s)$ the Laplace transform,

$$Y(s) = \mathcal{L}[y(t)] = \int_0^\infty e^{-st} y(t)dt.$$

(a) Find an expression for $Y(s)$ as a sum of ratios of polynomials in s.

(b) Determine the partial fraction expansion of $Y(s)$.

(c) Determine $y(t)$ by computing the inverse Laplace transform of $Y(s)$.

Problem 3 The general *skew-symmetric* real 2×2 matrix is,

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix},$$

where b is a real number. Prove that the eigenvalues of A of the form $\lambda = \pm i\mu$ for some real number μ. Find all values of b such that there is a single repeated eigenvalue.

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Problem 4 Let λ be a real number and let A be the following 3×3 matrix,

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Let a_1, a_2, a_3 be real numbers. Consider the following initial value problem,

$$\begin{cases} \mathbf{y}'(t) = A\mathbf{y}(t), \\ \mathbf{y}(0) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{cases}$$

Denote by $\mathbf{Y}(s)$ the Laplace transform of $\mathbf{y}(t)$, i.e.,

$$\mathbf{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix}, \quad Y_i(s) = \mathcal{L}[y_i(t)], \ i = 1, 2, 3.$$

(a) Express both $\mathcal{L}[\mathbf{y}'(t)]$ and $\mathcal{L}[A\mathbf{y}(t)]$ in terms of $\mathbf{Y}(s)$.

(b) Using part (a), find an equation that $\mathbf{Y}(s)$ satisfies, and iteratively solve the equation for $Y_3(s)$, $Y_2(s)$ and $Y_1(s)$, in that order.

(c) Determine $\mathbf{y}(t)$ by applying the inverse Laplace transform to $Y_1(s)$, $Y_2(s)$ and $Y_3(s)$.

Problem 5 For each of the following matrices A, compute the following,

(i) Trace(A),
(ii) $\det(A)$,
(iii) the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$,
(iv) the eigenvalues of A (both real and complex), and
(v) for each eigenvalue λ a basis for the space of λ-eigenvectors.

(a) The 2×2 matrix with real entries,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Hint: See Problem 3.

(b) The 3×3 matrix with real entries,

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$