18.034 PRACTICE EXAM 2, SPRING 2004

Problem 1 Let r be a positive real number. Consider the 2^{nd} order, linear differential equation,

$$y'' - \left(r + \frac{3}{t}\right)y' + \left(\frac{2r}{t} + \frac{3}{t^2}\right)y = 0,$$

where $y(t)$ is a function on $(0, \infty)$. One solution of this equation is $y_1(t) = te^{rt}$. Use Wronskian reduction of order to find a second solution $y_2(t)$.

Problem 2 An undamped harmonic oscillator satisfies the ODE,

$$y'' + \omega^2 y = 0.$$

Let $y(t)$ be a solution of this ODE for $t < \tau$. At some time $\tau > 0$, the oscillator is given an *impulse* of size $v > 0$. In other words, if

$$\begin{cases} \lim_{t \to \tau^-} y(t) &= y_0, \\ \lim_{t \to \tau^-} y'(t) &= v_0 \end{cases}$$

then for $t > \tau$, $y(t)$ is a solution of the IVP,

$$\begin{cases} y'' + \omega^2 y = 0, \\ y(\tau) = y_0, \\ y'(\tau) = v_0 + v \end{cases}$$

(a) Write $y(t)$ in normal form $A\cos(\omega t - \phi)$ for $t < \tau$, and in normal form $y(t) = B\cos(\omega t - \psi)$ for $t > \tau$. Find an equation expressing B^2 in terms of A^2, v_0 and v.

(b) If the goal of the impulse is to maximize the amplitude B, at what moment τ in the cycle of the oscillator should the impulse be applied? If the goal is minimize the amplitude B, at what moment τ should the impulse be applied?

Problem 3 Consider the following constant coefficient linear ODE,

$$y''' + y = 0.$$

(a) Find the characteristic polynomial and find all real and complex roots.

(b) Find the general *real-valued* solution of the ODE.

(c) Find a particular solution of the driven ODE,

$$y''' + y = \cos(\sqrt{3}t/2).$$

Problem 4 The linear ODE,

$$y'' + (t - 3/t)y' - 2y = 0,$$

has a basic solution pair $y_1(t) = e^{-\frac{t^2}{2}}$, $y_2(t) = t^2 - 2$.

(a) Find the Wronskian $W[y_1, y_2](t)$.

(b) Use variation of parameters to find a particular solution of the driven ODE,

$$y'' + (t - 3/t)y' - 2y = t^4.$$

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Problem 5 Recall that $PC_{\mathbb{R}}(0,1]$ is the set of all piecewise continuous real-valued functions on the interval $(0,1]$. The inner product on this set is,

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Define $f_0(t) = 1$. For each integer $n \geq 1$, define $f_n(t)$ to be the piecewise continuous function whose value on $(0, \frac{1}{2^n}]$ is -1, whose value on $(\frac{1}{2^n}, \frac{2}{2^n}]$ is $+1$, whose value on $(\frac{2}{2^n}, \frac{3}{2^n}]$ is -1, whose value on $(\frac{3}{2^n}, \frac{4}{2^n}]$ is $+1$, etc. In other words,

$$f_n(t) = \begin{cases} -1, & \frac{2k-2}{2^n} < t \leq \frac{2k-2}{2^n} \quad \text{for } k = 1, \dots, 2^{n-1}, \\ \\ +1, & \frac{2k-1}{2^n} < t \leq \frac{2k}{2^n} \quad \text{for } k = 1, \dots, 2^{n-1}. \end{cases}$$

(a) Compute the integrals $\langle f_m, f_n\rangle$ and use this to prove that $(f_0, f_1, \dots)$ is an orthonormal sequence. (**Hint**: If $n > m$, consider the integral of f_n over one of the subintervals $(\frac{a}{2^m}, \frac{a+1}{2^m}]$. What fraction of the time is f_n positive and what fraction of the time is it negative?)

(b) Compute the *generalized Fourier coefficient*,

$$\langle t, f_n(t)\rangle = \int_0^1 t f_n(t)dt.$$

Prove it equals $\frac{1}{2^{n+1}}$. This gives the generalized Fourier series,

$$t = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f_n(t).$$

(c) Rewrite the series above as,

$$t = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1 + f_n(t)}{2}.$$

What is the relationship of this equation to the binary expansion of the real number t?