18.034 PRACTICE EXAM 1, SPRING 2004

Problem 1 A holding tank for a chemical solution is run on a 2-hour cycle. During the first hour of the cycle, solution flows in at a rate of a liters/hour and no solution flows out of the tank. The concentration of chemical of the inflow is a constant q. During the second hour of the cycle, the thoroughly mixed solution flows out at a rate of a liters/hour and no solution flows into the tank. The volume of the tank V is so large that it can be considered as roughly constant through the whole process. The constants a and q are positive.

(a) Denote by $y(t)$ the quantity of chemical in the tank after t seconds ($t = 0$ corresponds to the beginning of the cycle). Find a first-order linear ODE in normal form that $y(t)$ satisfies.

(b) The quantity at time $t = 0$ is y_0. Solve the corresponding IVP for $y(t)$.

(c) In the steady state, the quantity at time $t = 2$ equals the quantity at time $t = 0$. Using this constraint, find the steady state solution of the IVP.

Problem 2 The following first-order ODE is exact,

$$2tyy' + (y^2 - 3t^2) = 0.$$

(a) Determine a non-constant function $H(t, y)$ that is constant along all solution curves of the ODE.

(b) The region R where the ODE is defined is $t > 0, y > 0$. Find the formula for the general solution in this region.

(c) For which choices of $y(1) > 1$ is it true that the solution of the corresponding IVP is defined for all $t > 0$?

Problem 3 The following first-order ODE in normal form is "homogeneous",

$$y' = \frac{3t^2 - y^2}{2ty}.$$

(a) Make the substitution, $y(t) = tu(t)$, $y' = tu' + u$, to change the ODE into a separable ODE for $u(t)$.

(b) Solve the separable ODE for $u(t)$. Then back-substitute to find the solution for $y(t)$. How does this compare to your solution in the last problem?

Problem 4 Consider the following IVP,

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases},$$

defined on the interval $(-1, 1)$. Using the first-approximation function $y_0(t) = 1$, find the first 2 Picard iterates, $y_1(t)$ and $y_2(t)$.

Problem 5 Consider the following first-order autonomous ODE,

$$y' = sin(\pi y).$$

(a) Determine all equilibrium solutions.

(b) Sketch the portion of the state line where $-2 - \epsilon < y < 2 + \epsilon$ (ϵ a positive number less than 1).

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(c) Give a rough sketch of the solution curves in the region $-2-\epsilon < y < 2+\epsilon$. Label all equilibrium solutions in this region. Between every pair of equilibrium solutions, sketch several solution curves.

(d) For all the equilibrium solutions of the ODE, say which are stable, which are unstable, and which are neither.

Problem 6 Consider the ODE,

$$y' = \frac{y}{t} - 2y^2,$$

defined on the region $t > 0$. Suppose that at $t_1 > 0$, a solution $y(t)$ is between the two curves $y = \frac{1}{t}$, $y = 0$. Show that for all $t > t_1$, the solution is trapped between these curves.

Extra credit Let $t_1 > 0$ and $y_1 > 0$. Prove that the solution of the IVP,

$$\begin{cases} y' = \frac{y}{t} - 2y^2, \\ y(t_1) = y_1 \end{cases}$$

is bounded as $t \to \infty$.