

18.034 EXAM 3
APRIL 26, 2004

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Instructions: *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, and calculators are not allowed. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Table of Laplace Transforms

	$y(t)$	$Y(s) = \mathcal{L}[y(t)]$
1.	$y^{(n)}(t)$	$s^n Y(s) - (y^{(n-1)}(0) + \dots + s^{n-1}y(0))$
2.	t^n	$n!/s^{n+1}$
3.	$t^n y(t)$	$(-1)^n Y^{(n)}(s)$
4.	$\cos(\omega t)$	$s/(s^2 + \omega^2)$
5.	$\sin(\omega t)$	$\omega/(s^2 + \omega^2)$
6.	$e^{at}y(t)$	$Y(s - a)$
7.	$y(at), a > 0$	$\frac{1}{a}Y(s/a)$
8.	$S(t - t_0)y(t - t_0), t_0 \geq 0$	$e^{-st_0}Y(s)$
9.	$\delta(t - t_0), t_0 \geq 0$	e^{-st_0}
10.	$(S(t)y) * (S(t)z)$	$Y(s)Z(s)$
11.	$y(t), y(t + T) = y(t)$	$\frac{1}{1 - e^{-sT}} \int_0^T e^{-st}y(t)dt$

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Problem 1: _____ /30

Problem 1(30 points) Consider the following inhomogeneous 2nd order linear differential equation,

$$\begin{cases} y'' - 3y' + 2y = e^{2t}, \\ y(0) = 0, \\ y'(0) = 0 \end{cases}$$

Denote by $Y(s)$ the Laplace transform,

$$Y(s) = \mathcal{L}[y(t)] = \int_0^{\infty} e^{-st}y(t)dt.$$

(a)(5 points) Taking the Laplace transform of both sides of the ODE, find an equation that $Y(s)$ satisfies.

Solution: First of all, $\mathcal{L}[y'(t)] = sY(s)$ and $\mathcal{L}[y''(t)] = s^2Y(s)$ because $y(0) = y'(0) = 0$. Secondly, $\mathcal{L}[e^{2t}] = 1/(s - 2)$. Therefore,

$$(s^2 - 3s + 2)Y(s) = \frac{1}{s - 2}.$$

(b)(15 points) Determine the partial fraction expansion of $Y(s)$.

Solution: The quadratic polynomial $s^2 - 3s + 2$ factors as $(s - 2)(s - 1)$. Therefore,

$$Y(s) = \frac{1}{(s - 1)(s - 2)^2}.$$

The partial fraction decomposition is of the form,

$$Y(s) = \frac{a}{s - 2} + \frac{b}{(s - 2)^2} + \frac{c}{s - 1}.$$

By the Heaviside cover-up method,

$$b = \frac{1}{(1 - 2)^2} = 1, \quad c = \frac{1}{(2 - 1)} = 1.$$

That leaves only a to determine. Substituting $s = 0$ in both sides of the equation gives,

$$\frac{1}{(-1)(-2)^2} = \frac{a}{(-2)} + \frac{1}{(-2)^2} + \frac{1}{(-1)},$$

i.e.,

$$\frac{a}{2} = \frac{1}{4} + \frac{1}{4} - 1 = -\frac{1}{2}.$$

Therefore $a = -1$. So the partial fraction decomposition is,

$$Y(s) = \frac{1}{(s - 1)(s - 2)^2} = \frac{-1}{s - 2} + \frac{1}{(s - 2)^2} + \frac{1}{s - 1}.$$

(c)(10 points) Determine $y(t)$ by computing the inverse Laplace transform of $Y(s)$.

Solution: The inverse Laplace transforms are,

$$\mathcal{L}^{-1}[1/(s - 1)] = e^t, \quad \mathcal{L}^{-1}[1/(s - 2)] = e^{2t}, \quad \mathcal{L}^{-1}[1/(s - 2)^2] = te^{2t}.$$

Therefore,

$$y(t) = -e^{2t} + te^{2t} + e^t.$$

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Problem 2: _____ /20

Problem 2(20 points) A real $n \times n$ matrix is *symmetric* if it is equal to its transpose, i.e. $A = A^T$. The general real symmetric 2×2 matrix is,

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

(a)(10 points) Let A be the real symmetric 2×2 matrix above. Compute the trace, determinant and characteristic polynomial of A .

Solution: The trace is $\text{Trace}(A) = a + d$, the determinant is $\det(A) = ad - b^2$, and the characteristic polynomial is,

$$p_A(\lambda) = \lambda^2 - \text{Trace}(A)\lambda + \det(A) = \lambda^2 - (a + d)\lambda + (ad - b^2).$$

(b)(10 points) Prove that every eigenvalue of the real symmetric 2×2 matrix A is real.

Solution: By the quadratic formula, the roots of $p_A(\lambda)$ are real iff the discriminant is nonnegative, i.e.

$$(-(a + d))^2 - 4(ad - b^2) \geq 0.$$

Simplifying, this is,

$$a^2 + 2ad + d^2 - 4ad + 4b^2 = a^2 - 2ad + d^2 + 4b^2 = (a - d)^2 + (2b)^2.$$

Since this is a sum of squares, it is nonnegative. So the eigenvalues of A are real. Moreover, it is clear that the eigenvalues are distinct unless $b = 0$ and $a = d$, i.e. unless A is a multiple of the identity matrix (but none of this was asked for in the problem).

Extra credit(15 points) Let A be a real symmetric $n \times n$ matrix, where n is arbitrary. Prove that every eigenvalue of A is real.

Let $\langle \mathbf{z}, \mathbf{w} \rangle$ be the Hermitian inner product on \mathbb{C}^n that assigns to every pair of vectors,

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},$$

the complex number,

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n.$$

Because A is a real symmetric matrix, for every pair \mathbf{z}, \mathbf{w} ,

$$\langle A\mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{z}, A\mathbf{w} \rangle.$$

Let λ be an eigenvalue of A , and let \mathbf{z} be a (nonzero) eigenvector for λ . Then,

$$\lambda \langle \mathbf{z}, \mathbf{z} \rangle = \langle \lambda \mathbf{z}, \mathbf{z} \rangle = \langle A\mathbf{z}, \mathbf{z} \rangle = \langle \mathbf{z}, A\mathbf{z} \rangle = \langle \mathbf{z}, \lambda \mathbf{z} \rangle = \bar{\lambda} \langle \mathbf{z}, \mathbf{z} \rangle.$$

Because \mathbf{z} is nonzero, $\langle \mathbf{z}, \mathbf{z} \rangle$ is nonzero. Therefore $\lambda = \bar{\lambda}$, i.e., λ is a real number.

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Problem 3: _____ /25

Problem 3(25 points) For each of the following matrices A , compute the following,

- (i) $\text{Trace}(A)$,
- (ii) $\det(A)$,
- (iii) the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$,
- (iv) the eigenvalues of A (both real and complex), and
- (v) for each eigenvalue λ a basis for the space of λ -eigenvectors.

(a)(15 points) The 2×2 matrix with real entries,

$$A = \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}.$$

Solution: First of all, $\text{Trace}(A) = -5$ and $\det(A) = 1(-6) - (-4)3 = -6 + 12 = 6$. Therefore the characteristic polynomial is,

$$p_A(\lambda) = \lambda^2 - \text{Trace}(A)\lambda + \det(A) = \lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2).$$

Therefore the eigenvalues are,

$$\lambda_1 = -3, \quad \lambda_2 = -2.$$

For $\lambda_1 = -3$, the eigenvectors are the (nonzero) nullvectors of $A + 3I$, i.e.,

$$\begin{bmatrix} 4 & 3 \\ -4 & -3 \end{bmatrix}.$$

Therefore, a basis for the nullvectors is,

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

For $\lambda = -2$, the eigenvectors are the (nonzero) nullvectors of $A + 2I$, i.e.,

$$\begin{bmatrix} 3 & 3 \\ -4 & -4 \end{bmatrix}.$$

Therefore, a basis for the nullvectors is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(b)(10 points) The 3×3 matrix with real entries,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: Because this is an upper triangular matrix, the trace, determinant and characteristic polynomial are easy to compute: $\text{Trace}(A) = 4$, $\det(A) = 2$, and the characteristic polynomial is,

$$p_A(\lambda) = (\lambda - 1)^2(\lambda - 2).$$

So the eigenvalues are, $\lambda_1 = 1$ (with multiplicity 2) and $\lambda_2 = 2$.

For $\lambda_1 = 1$, the eigenvectors are the (nonzero) nullvectors of $A - I$, i.e.,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore a basis for the nullvectors is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda_2 = 2$, the eigenvectors are the (nonzero) nullvectors of $A - 2I$, i.e.,

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Therefore a basis for the nullvectors is,

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Problem 4: _____ /25

Problem 4(25 points)

Let $f(t)$ be a function of exponential type, let $\mathbf{f}(t)$ be the following column vector,

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix},$$

and let A be the following 2×2 matrix,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Consider the following inhomogeneous system of 1st order linear ODEs,

$$\begin{cases} \mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{f}(t), \\ \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

Denote by $\mathbf{Y}(s)$ the Laplace transform of $\mathbf{y}(t)$, i.e.,

$$\mathbf{Y}(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}, \quad Y_i(s) = \mathcal{L}[y_i(t)], \quad i = 1, 2.$$

Denote by $F(s)$ the Laplace transform of $f(t)$.

(a)(5 points) Express $\mathcal{L}[\mathbf{y}'(t)]$, $\mathcal{L}[A\mathbf{y}(t)]$ and $\mathcal{L}[\mathbf{f}(t)]$ in terms of $\mathbf{Y}(s)$ and $F(s)$.

Solution: Because $\mathbf{y}(0) = 0$,

$$\mathcal{L}[\mathbf{y}'] = s\mathbf{Y}(s).$$

Because \mathcal{L} is a linear transformation,

$$\mathcal{L}[A\mathbf{y}(t)] = A\mathbf{Y}(s).$$

Finally,

$$\mathcal{L}[\mathbf{f}(t)] = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}.$$

(b)(10 points) Using part (a), find an equation that $\mathbf{Y}(s)$ satisfies, and iteratively solve the equation for $Y_2(s)$ and $Y_1(s)$, in that order.

Solution: The equation is,

$$s\mathbf{Y}(s) = A\mathbf{Y}(s) + \begin{bmatrix} 0 \\ F(s) \end{bmatrix}.$$

Writing this out,

$$\begin{cases} sY_1(s) &= Y_2(s) \\ sY_2(s) &= F(s) \end{cases}$$

Therefore $Y_2(s) = (1/s)F(s)$ and $Y_1(s) = (1/s)Y_2(s)$, i.e.,

$$\begin{cases} Y_1(s) &= (1/s^2)F(s) \\ Y_2(s) &= (1/s)F(s) \end{cases}$$

(c)(10 points) Determine $y_1(t)$ and $y_2(t)$ by applying the inverse Laplace transform to $Y_1(s)$ and $Y_2(s)$. Your answer should be expressed in terms of a convolution involving $f(t)$.

Solution: The inverse Laplace transform of $1/s$ is 1. The inverse Laplace transform of $1/s^2$ is t . Therefore,

$$\begin{cases} y_1(t) &= \mathcal{L}^{-1}[(1/s^2)F(s)] = (S(t)t) * (S(t)f(t)), \\ y_2(t) &= \mathcal{L}^{-1}[(1/s)F(s)] = (S(t)1) * (S(t)f(t)) \end{cases}$$

Written out in terms of integrals,

$$y_1(t) = \int_0^t (t-u)f(u)du, \quad y_2(t) = \int_0^t f(u)du.$$