18.034 EXAM 1
FEBRUARY 25, 2004

Name: ______________________________________

Problem 1:	________	/25
Problem 2:	________	/25
Problem 3:	________	/20
Problem 4:	________	/15
Problem 5:	________	/15
Total:	________	/100

Instructions: *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, and calculators are not allowed. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Date: Spring 2004.

Name: ________________________________ **Problem 1**: ______________ /25

Problem 1(25 points) Contaminated fluid flows into a large container with volume V at a rate of a liters/sec. The concentration of the contaminant is a constant q. Denote by $y(t)$ the total mass of contaminant in the container at time t. Chemical reactions neutralize the contaminant in the container at a rate $by(t)$, where b is a constant. Treated fluid flows out of the container at the rate of a liters/sec. The constants a, q and V are positive, and b is nonnegative.

(a)(10 points) Find a first-order linear ODE in normal form that $y(t)$ satisfies.

Solution The rate of flow of contaminant into the container is aq. There are 2 contributions to flow out of the container. Fluid flow out of the container removes contaminant at a rate $-a\frac{y}{V}$. And the chemical reaction removes contaminant at a rate $-by(t)$. By the balance law the ODE is,

$$y'(t) = aq - by(t) - \frac{a}{V}y(t).$$

The normal form of this first-order linear ODE is,

$$y' + \left(b + \frac{a}{V}\right) y = aq.$$

(b)(10 points) At time $t = 0$, the amount of contaminant in the container is y_0. Using an integrating factor, find $y(t)$ for $t > 0$.

Solution Denote $\lambda = b + \frac{a}{V}$. An integrating factor is $e^{\lambda t}$, which gives the ODE,

$$(e^{\lambda t}y)' = aqe^{\lambda t}.$$

Antidifferentiating both sides,

$$e^{\lambda t}y = \frac{aq}{\lambda}e^{\lambda t} + \left(y_0 - \frac{aq}{\lambda}\right).$$

Cancelling $e^{\lambda t}$ from both sides and plugging in for λ,

$$y(t) = qV\frac{1}{1+\frac{bV}{a}} + \left(y_0 - qV\frac{1}{1+\frac{bV}{a}}\right)e^{-(b+\frac{a}{V})t}.$$

(c)(5 points) In the steady-state, what is the concentration of contaminant in the outflowing fluid, i.e. what is $y(t)/V$? How does the steady-state concentration for $b > 0$ compare to the steady-state concentration for $b = 0$?

Solution The steady-state solution for $y(t)$ is,

$$y(t) = qV\frac{1}{1+\frac{bV}{a}}.$$

Therefore the steady-state concentration is,

$$\frac{y(t)}{V} = q\frac{1}{1+\frac{bV}{a}}.$$

The steady-state concentration for $b = 0$ is q, the same as the concentration in the inflowing fluid. So when $b > 0$, the concentration of chemical in the outflowing fluid as compared to the inflowing fluid is decreased by a factor $\frac{1}{1+\frac{bV}{a}}$.

Name: ______________________________ **Problem 2**: ____________ /25

Problem 2(25 points) Consider the first-order ODE,

$$(y-t)y' + (2t-y) = 0,$$

where $y = y(t)$. This ODE is exact.

(a)(10 points) Find a non-constant function $H(t,y)$ that is constant on all solution curves.

Solution The function $H(t,y)$ satisfies,

$$\frac{\partial H}{\partial t} = 2t - y, \quad \frac{\partial H}{\partial y} = y - t.$$

From the first equation, $H(t,y) = t^2 - ty + h(y)$ for some differentiable function $h(t)$. From the second equation, $-t + h'(y) = y - t$. So $h(y) = \frac{1}{2}y^2 + C$ for some constant C. Setting $C = 0$, the following is a function that is constant on all solution curves,

$$H(t,y) = \frac{1}{2}y^2 - ty + t^2 = \frac{1}{2}(y-t)^2 + \frac{1}{2}t^2.$$

(b)(10 points) Write down the equation of a general integral curve of this system. What part or parts of this integral curve are solution curves?

Solution The implicit equation of a general integral curve is,

$$(y-t)^2 + t^2 = C, \quad C \geq 0.$$

For $C = 0$ this is simply the point $(0,0)$ (we ignore this case). For $C > 0$, the curve is an ellipse. There are two solution curves on this integral curve,

$$\begin{cases} y_+(t) = t + \sqrt{C - t^2}, & -C \leq t \leq C \\ y_-(t) = t - \sqrt{C - t^2}, & -C \leq t \leq C \end{cases}$$

(c)(5 points) If the initial value $y(0) = y_0$ is positive, how much time elapses before the solution of the IVP is undefined?

Solution The point $(0, y_0)$ lies on the graph of the solution curve $y_+(t)$ for the constant $C = y_0^2$. The solution becomes undefined when $t^2 = C = y_0^2$, i.e. the elapsed time is y_0.

Name: ____ **Problem 3**: __________________ /20

Problem 3(20 points) The following first-order ODE in normal form is a *Bernoulli equation*,

$$y' = 2\frac{y}{t} - ty^2, \quad t > 0.$$

To simplify this ODE, substitute $u = 1/y$, $u' = -y'/y^2$.

(a)(15 points) Rewrite the ODE in terms of t and u, solve the resulting ODE, and back-substitute to find $y(t)$.

Solution The new ODE for u is,

$$u' = \frac{-y'}{y^2} = \frac{-2}{t}u + t.$$

This is a first-order linear ODE with normal form,

$$u' + \frac{2}{t}u = t.$$

An integrating factor is t^2,

$$(t^2u)' = t^3,$$

with solution,

$$t^2u = \frac{1}{4}t^4 + C, \text{ i.e. } u(t) = \frac{t^4 + C'}{4t^2}.$$

Back-substituting,

$$y(t) = \frac{1}{u} = \frac{4t^2}{t^4 + C'}.$$

(b)(5 points) Were any solutions lost by making the substitution? If so, say what they are.

Solution The solution is only valid if $y(t) \neq 0$. The constant function $y(t) \equiv 0$ is a solution that is lost by the substitution.

Name: ______________________________ **Problem 4**: ____________ /15

Problem 4(15 points) Consider the first-order ODE with initial condition,

$$\begin{cases} y' = y, \\ y(0) = 1 \end{cases}$$

(a)(5 points) For the first-approximation function, take $y_0(t) = 1$. Compute the first 2 Picard iterates, $y_1(t)$ and $y_2(t)$.

Solution By definition, the Picard iterates are recursively defined by

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds.$$

In this case,

$$y_1(t) = 1 + \int_0^t 1ds = 1 + t.$$

Applying this again gives,

$$y_2(t) = 1 + \int_0^t (1+s)ds = 1 + t + \frac{1}{2}t^2.$$

(b)(10 points) Find a formula for the n^{th} Picard iterate, $y_n(t)$. Justify your answer.

Solution The first 2 Picard iterates suggest that $y_n(t)$ is a polynomial of degree n,

$$y_n(t) = a_{n,0} + a_{n,1}t + \cdots + a_{n,k}t^k + \cdots + a_{n,n}t^n.$$

Plugging this, the formula for the next Picard iterate is,

$$y_{n+1}(t) = 1 + \int_0^t \sum_{k=0}^{n} a_{n,k}s^k ds = 1 + \sum_{k=0}^{n} \frac{a_{n,k}}{k+1}t^{k+1}.$$

This leads to the recursion relation, $a_{n+1,k+1} = \frac{a_{n,k}}{k+1}$, $a_{n,0} = 1$. The solution is,

$$a_{n,k} = \frac{1}{k!}, \text{ i.e. } y_n(t) = 1 + \sum_{k=1}^{n} \frac{1}{k!}t^k.$$

Of course this is consistent with the true solution, $y(t) = e^t$, since $y_n(t)$ is the n^{th} Taylor approximation of e^t.

Extra credit(5 points) Find the formula for the n^{th} Picard iterate associated to the IVP,

$$\begin{cases} y' = \cos(t), \\ y(0) = 0, \end{cases}$$

For first-approximation function, take $y_0(t) = 0$.

Solution Every Picard iterate leads to the formula,

$$y_n(t) = 0 + \int_0^t \cos(s)ds = \sin(t).$$

Name: ______________________________ **Problem 5**: ____________ /15

Problem 5(15 points) Consider the first-order, autonomous ODE,

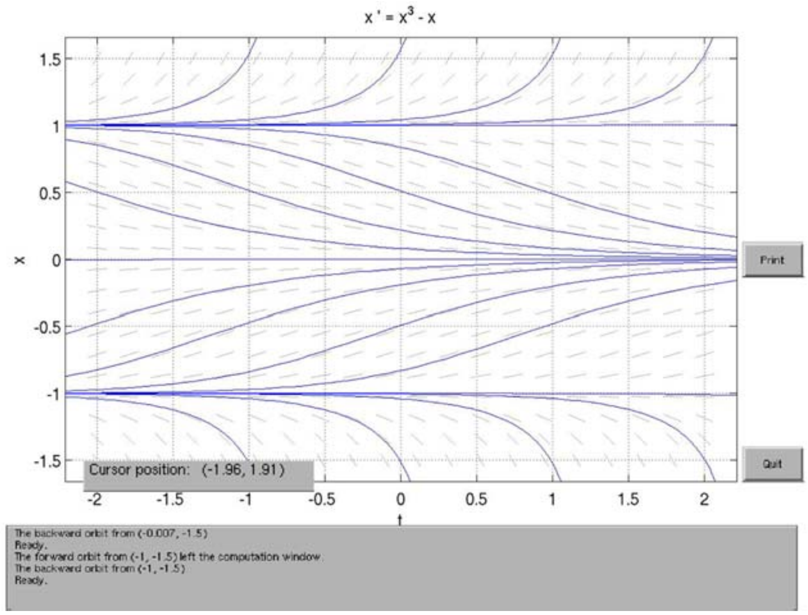
$$y' = y^3 - y.$$

(a)(10 points) Find all equilibrium solutions, sketch the state line, and sketch several solutions including all equilibrium solutions, and at least one solution between any 2 equilibrium solutions.

Solution The equilibrium solutions are the zeros of the rate function. The rate function factors as,

$$y^3 - y = y(y^2 - 1) = y(y-1)(y+1).$$

Thus there are three equilibrium solutions $y_1(t) = 1$, $y_2(t) = 0$ and $y_3(t) = 0$. For $t > 1$, say for $t = 2$, $(2)^3 - 2 = 6$ is positive. For $0 < t < 1$, say $t = \frac{1}{2}$, $\frac{1}{2^3} - \frac{1}{2} = -\frac{3}{8}$ is negative. For $-1 < t < 1$, say $t = \frac{-1}{2}$, $\frac{-1}{2^3} - \frac{-1}{2} = \frac{3}{8}$ is positive. For $t < -1$, say $t = -2$, $(-2)^3 - (-2) = -6$ is negative. An image file of the state line and several solution curves is given below.



(b)(5 points) For each equilibrium solution, state whether it is stable, unstable or neither.

Solution The equilibrium solution $y_1(t) = 1$ is unstable, the equilibrium solution $y_2(t) = 0$ is stable and the equilibrium solution $y_3(t) = -1$ is unstable.