18.034 SOLUTIONS TO PROBLEM SET 8

Due date: Friday, April 23 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

This problem set is essentially a reading assignment. I had originally intended to present the material in the problem set in lecture. However, this material is less relevant to other topics in this course, and there is no time to present it in lecture. Each student will receive 10 points simply for reading through this problem set. At several places, you are asked to work-through and write-up details in the derivation. You will turn in this write-up to be graded.

Problem 1(30 points) A collection of N identical particles of mass $m_0 = \cdots = m_{N-1} = m$ are allowed to oscillate about their equilibrium positions. Denote by $x_0(t), x_1(t), \dots, x_{N-1}(t)$ the displacement of the masses from equilibrium. The mass m_0 is connected to a motionless base by a spring. It is also connected to mass m_1 by a spring. For $i = 1, \dots, N-2$, mass m_i is connected to mass m_{i-1} and mass m_{i+1} by springs. Finally, mass m_{N-1} is connected to mass m_{N-2} and to a motionless base by a spring. Each spring is identical and has spring constant κ. Time is measured in units so that the frequency $\sqrt{\kappa/m}$ equals 1. The equations of motion for this system are,

$$\mathbf{x}'' = A_N \mathbf{x},\ \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix},$$

where A_N is the $N \times N$ matrix,

$$A_N = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}$$

In other words,

$$(A_N)_{i,j} = \begin{cases} 1, & j = i+1, \\ -2, & j = i, \\ 1, & j = i-1, \\ 0, & \text{otherwise} \end{cases}$$

In this problem, you will determine the general solution of this system of linear differential equations. Because of the special nature of this problem (namely, every eigenvalue of A has multiplicity 1), it is not necessary to reduce to a system of first-order linear differential equations.

(a), Step 1(10 points) In this step, you will find, for each integer n, all the roots of a polynomial p_n that is defined inductively below. Please read the whole derivation. In your writeup, *only fill in the missing steps in the third paragraph below*. You do not need to fill in the missing steps in the other paragraphs (but you are encouraged to work them out for yourself).

Let u be an indeterminate. Let $p_0, p_1, \ldots, p_n, \ldots$ be a sequence of real polynomials in u that satisfy the linear difference equation,

$$\left\{ \begin{array}{rl} p_{n+2} - 2up_{n+1} + p_n = 0, \\ p_0 = 1, \\ p_1 = 2u \end{array} \right. \tag{1}$$

where n varies among all nonnegative integers. In particular, $p_2 = 4u^2 - 1$, and $p_3 = 8u^3 - 4u$. (In fact, each p_n is a polynomial with integer coefficients.) In this step, you will prove that a closed formula for p_n is,

$$p_n = 2^n \prod_{k=1}^{n} (u - \cos(k\pi/(n+1))).$$

As a reality check, observe this gives,

$$\begin{array}{rcl} p_1 & = & 2(u - \cos(\pi/2)) = 2u \\ p_2 & = & 4(u - \cos(\pi/3))(u - \cos(2\pi/3)) = 4(u - 1/2)(u + 1/2) = 4u^2 - 1, \\ p_3 & = & 8(u - \cos(\pi/4))(u - \cos(2\pi/4))(u - \cos(3\pi/4)) = 8(u - 1/\sqrt{2})(u)(u + 1/\sqrt{2}) = 8u^3 - 4u \end{array}$$

Write up a careful proof of the missing steps in the next paragraph. To prove the formula, observe that it suffices to check when $|u| > 1$; two polynomials in u that agree for infinitely many values of u are equal (you don't have to write a proof of this). Assume that $p_n = C_+ r_+^n + C_- r_-^n$, where C_+, C_-, r_+, r_- are continuous functions in u that do not depend on n. For $r = r_+, r = r_-$, prove that r satisfies the *characteristic polynomial*,

$$r^2 - 2ur + 1 = 0.$$

Conclude that,

$$r_\pm = a \pm b, \ a = u, \ b = \sqrt{u^2 - 1}.$$

Plugging this in to the equations for p_0 and p_1, conclude that,

$$\left\{ \begin{array}{rcl} C_+ + C_- & = & 1, \\ (C_+ + C_-)a + (C_+ - C_-)b & = & 2a, \end{array} \right.$$

Solve this system of linear equations to get,

$$\left\{ \begin{array}{rcl} C_+ & = & (a+b)/2b, \\ C_- & = & -(a-b)/2b \end{array} \right.$$

Therefore one solution of Equation 1 for $|u| > 1$ is,

$$p_n = [(a+b)^{n+1} - (a-b)^{n+1}]/2b, \ a = u, \ b = \sqrt{u^2 - 1}.$$

But, of course, there is a unique solution: for each $n \geq 2$, p_n can be determined recursively in terms of p_{n-1} and p_{n-2}. Therefore, this is *the* solution of Equation 1.

Solution: Substituting in $p_n = C_+ r_+^n + C_- r_-^n$ to the finite difference equation yields,

$$C_+(r_+^2 - 2ur_+ + 1)r_+^n + C_-(r_-^2 - 2ur_- + 1)r_-^n = 0,$$

for every $n \geq 0$. Clearly, if $r_\pm^2 - 2ur_\pm + 1 = 0$, this equation is satisfied. This is all that is required for the rest of the argument: the strategy is to find one solution of the finite difference equation, not every solution (in the last step, we observe there is a unique solution). However, it is reasonable to check that every solution of the finite difference equation of the form $p_n = C_+ r_+^n + C_- r_-^n$ does satisfy $r_\pm^2 - 2ur_\pm + 1 = 0$. Suppose that $r_+^2 - 2ur_+ + 1 \neq 0$. Plugging in $n = 0, 1$ above gives,

$$C_-(r_-^2 - 2ur_- + 1) = -C_+(r_+^2 - 2ur_+ 1),$$
$$C_+(r_+^2 - 2ur_+ + 1)(r_+ - r_-) = 0$$

Therefore either $C_+ = 0$ or $r_- = r_+$ (or both). If $C_+ = 0$, then the conditions $p_0 = 1, p_1 = 2u$ imply that $C_- = 1$ and $r_- = 2u$. But then $p_2 = 4u^2$ which contradicts that $p_2 = 4u^2 - 1$. If $r_- = r_+$, then $p_n = (C_+ + C_-)r_+^n = p_0 r_+^n = r_+^n$. Therefore $r_+ = 2u$. But then $p_2 = 4u^2$ which again contradicts that $p_2 = 4u^2 - 1$. This contradiction proves that $r_+^2 - 2ur_+ + 1 = 0$. A similar argument proves that $r_-^2 - 2ur_- + 1 = 0$. This argument also proves that $r_+ \neq r_-$.

Because $r_+ \neq r_-$ and both are solutions of $r^2 - 2ur + 1$,

$$r_\pm = a \pm b,\ a = u,\ b = \sqrt{u^2 - 1},$$

up to relabeling r_+ and r_-. For any choice of C_+ and C_-, the sequence $p_n = C_+ r_+^n + C_- r_-^n$ satisfies the finite difference equation $p_{n+2} - 2up_{n+1} + p_n = 0$. The conditions $p_0 = 1$ and $p_1 = 2u$ imply that,

$$\begin{cases} C_+ + C_- &= 1, \\ (C_+ + C_-)a + (C_+ - C_-)b &= 2a, \end{cases}$$

In particular, $(C_+ - C_-)b = a$. Therefore,

$$\begin{cases} (C_+ + C_-)b &= b, \\ (C_+ - C_-)b &= a, \end{cases}$$

Solving, $2bC_+ = a + b$ and $2bC_- = -(a - b)$, i.e.,

$$\begin{cases} C_+ &= (a+b)/2b, \\ C_- &= -(a-b)/2b \end{cases}$$

For this choice of C_+ and C_-, p_n satisfies all three conditions. Therefore one solution is,

$$p_n = C_+ r_+^n + C_- r_-^n = [(a+b)^{n+1} - (a-b)^{n+1}]/2b, \quad a = u, \quad b = \sqrt{u^2 - 1}.$$

End of the solution

For the rest of this part, do not write up the missing details. The equation above also makes sense and is correct if $|u| < 1$, where $b = \sqrt{u^2 - 1}$ is interpreted as a complex number. Let $|u| < 1$. Then $p_n(u) = 0$ iff $(a+b)^{n+1} = (a-b)^{n+1}$. Both $a + b$ and $a - b$ are nonzero (because the imaginary part of each complex number is nonzero). Therefore $(a+b)^{n+1} = (a-b)^{n+1}$ iff one of the following equations holds,

$$(a + b) = \zeta_k(a - b),\ \zeta = e^{i2\pi k/(n+1)},$$

as k varies among the integers $0, \ldots, n$. Of course for $k = 0$, the equation is $a + b = a - b$, i.e. $b = 0$. Since $b \neq 0$, this case is ruled out.

Let $k = 1, \ldots, n$. If $(a + b) = \zeta_k(a - b)$, then

$$(\zeta_k - 1)a = (\zeta_k + 1)b.$$

Squaring both sides,

$$(\zeta_k - 1)^2 a^2 = (\zeta_k + 1)^2 b^2,$$

i.e.,

$$(\zeta_k - 1)^2 u^2 = (\zeta_k + 1)^2 (u^2 - 1).$$

Solving for u_2 gives,

$$2(2\zeta_k)u^2 = \zeta_k^2 + 2\zeta_k + 1,$$

i.e.,

$$u^2 = (\zeta_k + \zeta_k^{-1} + 2)/4.$$

Simplifying,

$$\zeta_k + \zeta_k^{-1} = e^{i2\pi k/(n+1)} + e^{-i2\pi k/(n+1)} = 2\cos(2\pi k/(n+1)).$$

Also, $2 + 2\cos(\theta) = 4\cos^2(\theta/2)$. Therefore, $u^2 = \cos^2(\pi k/(n+1))$. So $u = \pm\cos(\pi k/(n+1))$. But of course $-\cos(\pi k/(n+1)) = \cos(\pi(n+1-k)/(n+1))$. Therefore, every root of $p_n(u) = 0$ with $|u| < 1$ is of the form $\cos(\pi k/(n+1))$ for some integer $k = 1, \dots, n$.

Reversing the steps above, $\cos(\pi k/(n+1))$ is a root of p_n for every $k = 1, \dots, n$. Also, for $0 < \theta < \pi$, the function $\cos(\theta)$ is strictly decreasing. Therefore the real numbers $\cos(\pi k/(n+1))$ are all distinct. This gives n distinct real roots of the degree n polynomial p_n. Since a polynomial of degree n has at most n real roots, counted with multiplicity, *every* root of p_n is of the form $\cos(\pi k/(n+1))$, and each of these roots has multiplicity 1. It is straightforward to compute that the leading coefficient of p_n is 2^n. Therefore,

$$p_n = 2^n \prod_{k=1}^{n} (u - \cos(\pi k/(n+1))).$$

(b), Step 2(10 points) For each integer $N \geq 1$, define $P_N(\lambda)$ to be the characteristic polynomial $\det(\lambda I_{N\times N} - A_N)$. Define $P_0(\lambda) = 1$. **Using cofactor expansion along the first row, prove that the sequence of polynomials $P_0, P_1, P_2, \dots$ satisfies Equation 1 where $u = \frac{\lambda}{2} + 1$.**

Solution: By direct computation, $P_0 = 1$ and $P_1 = \lambda + 2$. Let $N \geq 0$ and consider P_{N+2}. By cofactor expansion along the first row,

$$P_{N+2} = \det \begin{bmatrix} \lambda+2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda+2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \lambda+2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \lambda+2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda+2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & \lambda+2 \end{bmatrix}$$

equals,

$$(\lambda+2)\det \begin{bmatrix} * & * & * & * & \dots & * & * \\ * & \lambda+2 & -1 & 0 & \dots & 0 & 0 \\ * & -1 & \lambda+2 & -1 & \dots & 0 & 0 \\ * & 0 & -1 & \lambda+2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & 0 & 0 & \dots & \lambda+2 & -1 \\ * & 0 & 0 & 0 & \dots & -1 & \lambda+2 \end{bmatrix}$$

$$+1\det \begin{bmatrix} * & * & * & * & \dots & * & * \\ -1 & * & -1 & 0 & \dots & 0 & 0 \\ 0 & * & \lambda+2 & -1 & \dots & 0 & 0 \\ 0 & * & -1 & \lambda+2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & 0 & 0 & \dots & \lambda+2 & -1 \\ 0 & * & 0 & 0 & \dots & -1 & \lambda+2 \end{bmatrix}$$

where the asterisks are just placeholders for then entries to be deleted in computing the determinant. The first determinant above, is simply P_{N+1}. For the second determinant, perform cofactor expansion along the first column (which has only 1 nonzero entry),

$$\det\begin{bmatrix} * & * & * & * & \dots & * & * \\ -1 & * & -1 & 0 & \dots & 0 & 0 \\ 0 & * & \lambda+2 & -1 & \dots & 0 & 0 \\ 0 & * & -1 & \lambda+2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & 0 & 0 & \dots & \lambda+2 & -1 \\ 0 & * & 0 & 0 & \dots & -1 & \lambda+2 \end{bmatrix} =$$

$$-1\det\begin{bmatrix} * & * & * & * & \dots & * & * \\ * & * & * & * & \dots & * & * \\ * & * & \lambda+2 & -1 & \dots & 0 & 0 \\ * & * & -1 & \lambda+2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & 0 & 0 & \dots & \lambda+2 & -1 \\ * & * & 0 & 0 & \dots & -1 & \lambda+2 \end{bmatrix}$$

This is $-P_N$. Therefore, $P_{N+2} = (\lambda+2)P_{N+1} - P_N$. Defining $u = 1 + \frac{\lambda}{2}$, this is,

$$\begin{cases} P_{N+2} - 2uP_{N+1} + P_N = 0, \\ P_0 = 1, \\ P_1 = 2u \end{cases}$$

End of the solution

It then follows that the characteristic polynomial of A_N is,

$$P_N(\lambda) = \prod_{k=1}^{N} (\lambda + 2(1 - c_k)),\ c_k = \cos(\pi k/(N+1)).$$

Therefore the eigenvalues of A_N are $\lambda_k = -2 + 2c_k = -(2\sin(\pi k/(N+1)))^2$, for $k = 1, \dots, N$.

(c), Step 3(10 points) Let the integer $N \geq 1$ be fixed. For each integer $k = 1, \dots, N$, consider the eigenvalue λ_k. Denote,

$$\mathbf{x}_{(k)} = \begin{bmatrix} x_{(k),0} \\ x_{(k),1} \\ \vdots \\ x_{(k),N-1} \end{bmatrix}$$

the eigenvector of A_N with eigenvalue λ_k that is normalized by the condition $x_{(k),0} = 1$.

Prove that the sequence $x_{(k),0}, x_{(k),1}, \dots$ satisfies Equation 1 where $u = c_k$. It then follows that for each $n = 0, \dots, N-1$, the entry $x_{(k),n}$ is given by,

$$x_{(k),n} = 2^n \prod_{l=1}^{n} \left(\cos(\pi k/(N+1)) - \cos(\pi l/(n+1))\right).$$

Solution: Because $\mathbf{x}_{(k)}$ is an eigenvector with eigenvalue λ_k, $A\mathbf{x}_{(k)} = \lambda_k \mathbf{x}_{(k)}$. Expanding this out,

$$\begin{cases} & & (-2)x_{(k),0} & + & x_{(k),1} & = & \lambda_k x_{(k),0}, \\ x_{(k),n} & + & (-2)x_{(k),n+1} & + & x_{(k),n+2} & = & \lambda_k x_{(k),n+1}, \quad n = 0, \dots, N-3, \\ x_{(k),N-2} & + & (-2)x_{(k),N-1} & & & = & \lambda_k x_{(k),N-1} \end{cases}$$

In particular, $x_{(k),0} = 1$ by definition, and the first equation gives $x_{(k),1} = 2 + \lambda_k = 2c_k$. Therefore, for $n = 0, \dots, N-3$, we have,

$$\begin{cases} x_{(k),n+2} - 2c_k x_{(k),n+1} + x_{(k),n} = 0, \\ x_{(k),0} = 1, \\ x_{(k),1} = 2c_k \end{cases}$$

This is Equation 1 with $u = c_k$. Therefore, for $n = 0, \dots, N-1$,

$$x_{(k),n} = 2^n \prod_{l=1}^{n} \left(\cos(\pi k/(N+1)) - \cos(\pi l/(n+1)) \right).$$

As a double-check, observe that we also have $-2c_k x_{(k),N-1} + x_{(k),N-2} = 0$. This is equivalent to the equation $x_{(k),N} = 0$, where $x_{(k),N}$ is defined by the product formula above. In the product formula, the factor for $l = k$ is $\cos(\pi k/(N+1)) - \cos(\pi k/(N+1)) = 0$. Thus $x_{(k),N} = 0$, as it must.

End of the solution

(d), Step 4(0 points) Having computed the eigenvalues λ_k and an eigenvector $\mathbf{x}_{(k)}$, it is now straightforward to solve the linear system of differential equations. Let k be an integer, $k = 1, \dots, N$. Let $z_k(t)$ be a real-valued function. Then $z_k(t)x_{(k)}$ is a solution of the linear system iff,

$$z_k'(t)\mathbf{x}_{(k)} = A(z_k(t)\mathbf{x}_{(k)}) = z_k(t)A\mathbf{x}_{(k)} = z_k(t)\lambda_k\mathbf{x}_{(k)}.$$

Therefore $z_k(t)\mathbf{x}_{(k)}$ is a solution iff,

$$z_k'(t) = \lambda_k z_k(t) = -(2\sin(\pi k/(N+1)))^2 z_k(t).$$

Denote $\omega_k = 2\sin(\pi k/(N+1))$. The general solution of this differential equation is,

$$z_k(t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t).$$

Therefore the general solution of the linear system of differential equations is,

$$x(t) = \sum_{k=1}^{N} (A_k \cos(\omega_k t) + B_k \sin(\omega_k t))\mathbf{x}_{(k)}, \ \omega_k = 2\sin(\pi k/(N+1)),$$

$$\mathbf{x}_{(k)} = \left[x_{(k),0}, \dots, x_{(k),N-1}\right]^{\mathrm{T}}, \ x_{(k),n} = 2^n \prod_{l=1}^{n} \left(\cos(\pi k/(N+1)) - \cos(\pi l/(n+1)) \right).$$

Problem 2(10 points) The general linear system of first order differential equations *not necessarily with constant coefficients* is,

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t).$$

Apart from the existence and uniqueness theorem proved in the beginning of the semester, there is little we can say about the solution of this differential equation. However, if $A(t)$ has a special form, sometimes we can get an equation for the solution that is a bit more explicit. In this problem you will deduce such a result.

(a), Step 1(5 points). Let $g(\lambda, t)$ be a C^∞ function defined on the (λ, t)-plane. Let S be an $n \times n$ real matrix. Assume that the eigenvalues of S, $\lambda_1, \dots, \lambda_k$ are all real and that none of the eigenspace is deficient, i.e. for $i = 1, \dots, k$, the generalized eigenspace $V_{\lambda_i}^{\text{gen}}$ equals the usual eigenspace V_{λ_i}.

Define $g(S, t)$ to be the unique matrix such that for each $i = 1, \dots, k$, $g(S, t)$ preserves the eigenspace V_{λ_i} and the restriction of $g(S, t)$ to this subspace equals $g(\lambda_i, t)$ times the identity matrix.

Let $\mathbf{v}$ be an n-vector and consider $g(S,t)\mathbf{v}$. **Write up the proof of the following fact.** The function of t, $g(S,t)\mathbf{v}$ is C^∞ and for every nonnegative integer r,

$$\frac{d^r}{dt^r}\left(g(S,t)\mathbf{v}\right) = \left(\frac{\partial^r}{\partial t^r}g\right)(S,t)\mathbf{v}.$$

Solution: It suffices to prove that for some choice of ordered basis $\mathcal{B}$, every entry of the matrix $A = [g(S,t)]_{\mathcal{B},\mathcal{B}}$ is a C^∞ function, and $\frac{d^r}{dt^r}(A_{i,j})$ is the (i,j)-entry of $[(\frac{\partial^r}{\partial t^r}g)(S,t)]_{\mathcal{B},\mathcal{B}}$.

Because S is diagonalizable, there exists an ordered basis $\mathcal{B} = (v_1,\dots,v_n)$ such that each v_j is an eigenvector for S. For $j = 1,\dots,n$, define μ_j by $Sv_j = \mu_j v_j$. By definition, $[g(S,t)]_{\mathcal{B},\mathcal{B}}$ is a diagonal matrix whose (j,j)-entry is $g(\mu_j,t)$. Therefore every entry is a C^∞ function, and

$$\frac{d^r}{dt^r}g(\mu_j,t) = \frac{\partial^r g}{\partial t^r}(\mu_j,t),$$

which is the (j,j)-entry of the diagonal matrix $[(\frac{\partial^r}{\partial t^r}g)(S,t)]_{\mathcal{B},\mathcal{B}}$.

End of the solution

(b), Step 2(5 points) Next, let A be an $n\times n$ real matrix whose eigenvalues are all real, but such that the eigenspaces of A might be deficient. Denote by $\lambda_1,\dots,\lambda_k$ the eigenvalues of A. For each $i = 1,\dots,k$, denote by $V_{\lambda_i}^{\text{gen}}$ the generalized eigenspace of A. Define S to be the unique matrix such that for each $i = 1,\dots,k$, S preserves $V_{\lambda_i}^{\text{gen}}$ and the restriction of S to $V_{\lambda_i}^{\text{gen}}$ is λ_i times the identity matrix. Define $N = A - S$. Then S commutes with N, the matrix S is diagonalizable, and the matrix N is nilpotent, i.e. $N^{e+1} = 0$ for some integer e.

Define $g(A,t)$ to be the matrix,

$$g(A,t) = \sum_{m=0}^{e}\frac{1}{m!}N^m\left(\frac{\partial^m}{\partial\lambda^m}g\right)(S,t).$$

Let $\mathbf{v}$ be an n-vector and consider $g(A,t)\mathbf{v}$. **Write up the proof of the following fact.** The function of t, $g(A,t)\mathbf{v}$ is C^∞ and for every nonnegative integer r,

$$\frac{d^r}{dt^r}\left(g(A,t)\mathbf{v}\right) = \left(\frac{\partial^r}{\partial t^r}g\right)(A,t)\mathbf{v}.$$

(**Hint**: Use the equality of mixed partial derivatives, and reduce to part (a)).

Solution: First of all, by part (a), each of the functions $w_m(t) = (\frac{\partial^m}{\partial\lambda^m}g)(S,t)v$ is a C^∞ function. Therefore each $(1/m!)N^m w_m(t)$ is a C^∞ function (after all, N^m is a constant matrix). Thus the finite sum $g(A,t)v$ is a C^∞ function.

The equation,

$$\frac{d^r}{dt^r}\left(g(A,t)\mathbf{v}\right) = \left(\frac{\partial^r}{\partial t^r}g\right)(A,t)\mathbf{v},$$

is proved by induction on the positive integer r. Let $r = 1$. By part (a), each of the functions $(\frac{\partial^m}{\partial\lambda^m}g)(S,t)v$ is C^∞ and,

$$\frac{d}{dt}\left(\frac{\partial^m}{\partial\lambda^m}g\right)(S,t)v = \left(\frac{\partial}{\partial t}\left(\frac{\partial^m}{\partial\lambda^m}g\right)\right)(S,t)v.$$

Because g is C^∞, mixed partial derivatives are equal, i.e.

$$\frac{d}{dt}\left(\frac{\partial^m}{\partial\lambda^m}g\right)(S,t)v = \left(\frac{\partial^m}{\partial\lambda^m}\left(\frac{\partial g}{\partial t}\right)\right)(S,t)v.$$

Therefore,

$$\frac{d}{dt}(g(A,t)\mathbf{v}) = \sum_{m=1}^{e} \frac{1}{m!} N^m \left(\frac{\partial^m}{\partial \lambda^m} \left(\frac{\partial g}{\partial t} \right) \right) (S,t)v = \left(\frac{\partial g}{\partial t} \right) (A,t)\mathbf{v}.$$

This is the case $r = 1$ in the induction above.

By way of induction, suppose that $r > 1$ and that the result is proved for $r-1$. Define $h(\lambda, t) = \frac{\partial g}{\partial t}$. Then, by the induction hypothesis,

$$\frac{d^{r-1}}{dt^{r-1}} h(A,t)\mathbf{v} = \left(\frac{\partial^{r-1} h}{\partial t^{r-1}} \right) (A,t)\mathbf{v}.$$

Therefore,

$$\frac{d^r}{dt^r} g(A,t)\mathbf{v} = \frac{d^{r-1}}{dt^{r-1}} \frac{d}{dt} g(A,t)\mathbf{v} = \frac{d^{r-1}}{dt^{r-1}} \left(\frac{\partial g}{\partial t} \right) (A,t)\mathbf{v} =$$
$$\frac{d^{r-1}}{dt^{r-1}} h(A,t)\mathbf{v} = \left(\frac{\partial^{r-1} h}{\partial t^{r-1}} \right) (A,t)\mathbf{v} = \left(\frac{\partial^r g}{\partial t^r} \right) (A,t)\mathbf{v}.$$

The second equality follows from the case $r = 1$ and the fourth equality follows from the induction hypothesis. Therefore the result is proved for r. So the result is proved by induction on r.

End of the solution

(c), Step 3(0 points) Let $g(\lambda, t)$ and $h(\lambda, t)$ be C^∞ functions on the (λ, t)-plane. Let A be an $n \times n$ matrix all of whose eigenvalues are real. Then $g(S,t)$, $g(A,t)$, $h(S,t)$ and $h(A,t)$ all commute with each other. Moreover,

$$g(S,t)h(S,t) = (g \cdot h)(S,t), \text{ and } g(A,t)h(A,t) = (g \cdot h)(A,t).$$

To prove that $g(S,t)$ and $h(S,t)$ commute with each other, that they commute with $g(A,t)$ and $h(A,t)$, and that $g(S,t)h(S,t) = (g \cdot h)(S,t)$, it suffices to prove this after restricting to $V_{\lambda_i}^{\text{gen}}$ for every $i = 1, \ldots, k$. But for any C^∞ function $f(\lambda, t)$, the restriction of $f(S,t)$ to $V_{\lambda_i}^{\text{gen}}$ is simply $f(\lambda_i, t)$ times the identity matrix. Denote this by $f(S,t)_i$. Therefore $f(S,t)_i$ commutes with *every* linear operator on $V_{\lambda_i}^{\text{gen}}$. In particular, $g(S,t)_i$ and $h(S,t)_i$ commute with each other and with $g(A,t)_i$ and $h(A,t)_i$. Also, $g(S,t)_i \cdot h(S,t)_i$ is simply $g(\lambda_i, t) \cdot h(\lambda_i, t)$ times the identity matrix. This is the same as $(g \cdot h)(S,t)_i$. Therefore $g(S,t) \cdots h(S,t) = (g \cdot h)(S,t)$.

To prove that $g(A,t)$ and $h(A,t)$ commute with each other and that $g(A,t)h(A,t) = (g \cdot h)(A,t)$, it suffices to prove this after restricting to $V_{\lambda_i}^{\text{gen}}$ for every $i = 1, \ldots, k$. Denote by S_i and N_i the restrictions of S and N respectively to $V_{\lambda_i}^{\text{gen}}$. By definition,

$$g(A,t)_i h(A,t)_i = \sum_{l=0}^{e} \frac{1}{l!} N_i^l \left(\frac{\partial^l}{\partial \lambda^l} g \right) (S,t)_i \cdot \sum_{m=0}^{e} \frac{1}{m!} N_i^m \left(\frac{\partial^m}{\partial \lambda^m} h \right) (S,t)_i.$$

All of the matrices in this last sum commute. Therefore we may rearrange the (finite) sum in whatever order we wish,

$$g(A,t)_i h(A,t)_i = \sum_{l=0}^{e} \sum_{m=0}^{e} \frac{1}{l!} \frac{1}{m!} N_i^{l+m} (\frac{\partial^l g}{\partial \lambda^l} \cdot \frac{\partial^m h}{\partial \lambda^m})(S,t)_i.$$

Define $p = l + m$. For $p > e$, $N^p = 0$. Therefore the sum equals,

$$g(A,t)_i h(A,t)_i = \sum_{p=0}^{e} \frac{1}{p!} N_i^p \left(\sum_{m=0}^{p} \binom{p}{m} \frac{\partial^{p-m} g}{\partial \lambda^{p-m}} \cdot \frac{\partial^m h}{\partial \lambda^m} \right) (S,t)_i.$$

But of course,

$$\frac{\partial^p (gh)}{\partial \lambda^p} = \sum_{m=0}^{p} \binom{p}{m} \frac{\partial^{p-m} g}{\partial \lambda^{p-m}} \cdot \frac{\partial^m h}{\partial \lambda^m}.$$

Hence,

$$g(A,t)_i h(A,t)_i = \sum_{p=0}^{e} \frac{1}{p!} N^p \left(\frac{\partial^p (gh)}{\partial \lambda^p} \right) (S,t)_i = (g \cdot h)(A,t)_i.$$

Therefore $g(A,t) \cdot h(A,t) = (g \cdot h)(A,t)$. Because $g \cdot h = h \cdot g$, it follows that $g(A,t)$ commutes with $h(A,t)$.

(d), Step 4(0 points) Let $\mathcal{L}_\lambda$ be a linear differential operator of order $n+1$ in t,

$$\mathcal{L}_\lambda = \frac{\partial^{n+1}}{\partial t^{n+1}} + a_n(\lambda,t) \frac{\partial^n}{\partial t^n} + \cdots + a_1(\lambda,t) \frac{\partial}{\partial t} + a_0(\lambda,t),$$

where each of the functions $a_j(\lambda,t)$ is a C^∞ function. Let $y(\lambda,t)$ be a C^∞ function that is a solution of $\mathcal{L}_\lambda y(\lambda,t) = 0$.

Let A be a real $n \times n$ matrix all of whose eigenvalues are real. Let $\mathbf{v}$ be any n-vector. Then the function $\mathbf{y}(t) = y(A,t)\mathbf{v}$ is a C^∞ function in t that is a solution of the linear system of differential equations,

$$\mathcal{L}_A \mathbf{y}(t) = 0,$$

i.e.,

$$\frac{d^{n+1}}{dt^{n+1}} \mathbf{y}(t) + a_n(A,t) \frac{d^n}{dt^n} \mathbf{y}(t) + \cdots + a_0(A,t)\mathbf{y}(t) = 0.$$

This follows immediately from Steps 1–3.

(e), Step 5(0 points) Let t_0 be a real number. For $s \geq t_0$, let $\widetilde{K}(\lambda; s,t)$ be the unique solution of the initial value problem,

$$\begin{cases} \mathcal{L}_\lambda \widetilde{K}(\lambda; s,t) = 0, \\ \widetilde{K}(\lambda; s,s) = 0, \\ \vdots \\ \frac{\partial^{n-1}}{\partial t^{n-1}} \widetilde{K}(\lambda; s,s) = 0, \\ \frac{\partial^n}{\partial t^n} \widetilde{K}(\lambda; s,s) = 1 \end{cases}$$

Assume that $\widetilde{K}(\lambda; s,t)$ is a C^∞ function in all 3 variables λ, s and t.

Let $\mathbf{f}(t)$ be an n-vector whose entries are continuous functions of t. By the same arguments as in the handout on Green's functions, the unique solution of the IVP,

$$\begin{cases} \mathcal{L}_A \mathbf{y}(t) = \mathbf{f}(t), \\ \mathbf{y}(t_0) = 0, \\ \vdots \\ \mathbf{y}^{(n)}(t_0) = 0 \end{cases}$$

is given by,

$$\mathbf{y}(t) = \int_{t_0}^{t} \widetilde{K}(A; s,t) \mathbf{f}(s) ds,$$

where the integration is done entry-by-entry.

(f), Step 6(0 points) This is somewhat beside the point, but if $g(\lambda, t)$ is an analytic function in λ whose expansion about the origin is,

$$g(\lambda, t) = \sum_{r=0}^{\infty} c_r(t)\lambda^r,$$

and if all the eigenvalues of A are within the radius of convergence of this power series, then the following series converges to $g(A, t)$,

$$\sum_{r=0}^{\infty} c_r(t)A^r.$$

In particular, this holds if $g(\lambda, t)$ is a polynomial in λ or if $g(\lambda, t) = e^{\lambda t}$.