18.034 SOLUTIONS TO PROBLEM SET 6

Due date: Friday, April 9 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Each of the following problems is from the textbook. The point value of the problem is next to the problem.

(1)(10 points) p. 313, Problem 19

Solution: The problem is to solve the IVP,

$$\begin{cases} y'' - 2y' + 2y = 0, \\ y(0) = 0, \\ y'(0) = 1, \end{cases}$$

Denote $Y(s) = \mathcal{L}[y(t)]$. Then $\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s)$. And $\mathcal{L}[y''(t)] = s^2Y(s) - (y'(0) + sy(0)) = s^2Y(s) - 1$. This leads to the equation,

$$(s^2 - 2s + 2)Y(s) - 1 = \mathcal{L}[y'' - 2y' + 2y] = 0.$$

Therefore,

$$Y(s) = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1}.$$

By Item (11) on the Laplace transform table of the back cover,

$$\mathcal{L}^{-1}[1/(s^2 + 1)] = \sin(t).$$

By Item (26) on the table,

$$\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t).$$

Therefore,

$$\mathcal{L}^{-1}[1/((s-1)^2 + 1)] = e^t \sin(t).$$

So the solution of the IVP is $y(t) = e^t \sin(t)$.

(2)(5 points) p. 327, Problem 7

Solution: The inverse Laplace transform of $1/s^2$ is t. The inverse Laplace transform of $1/(s+1)$ is e^{-t}. Therefore the inverse Laplace transform is,

$$\mathcal{L}^{-1}[1/(s^2(s+1))] = (S(t)t) * (S(t)e^{-t}) = \int_0^t ue^{-(t-u)}du,$$

where, as usual, $S(t)$ is the step function.

This is not called for, but it is straightforward to compute that,

$$\frac{1}{s^2(s+1)} = \frac{1}{s+1} + \frac{1}{s^2} - \frac{1}{s}.$$

Therefore, the inverse Laplace transform is $e^{-t}+t-1$. Of course it is also straightforward to evaluate the anti-derivative above, and it also gives $e^{-t} + t - 1$.

(3)(10 points) p. 327, Problem 15

Solution: We want to compute the inverse Laplace operator of,

$$Y(s) = \frac{1}{s^2 - 1} = \frac{1}{(s-1)(s+1)}.$$

Denote the partial fraction expansion by,

$$\frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}.$$

By the Heaviside coverup method, $A = 1/2$ and $B = -1/2$. Therefore,

$$\frac{1}{(s-1)(s+1)} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}.$$

Therefore the inverse Laplace transform is,

$$\mathcal{L}^{-1}[Y(s)] = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t).$$

(4)(5 points) p. 335, Problem 3

We want to find the solution $y_\delta(t,u)$ of the IVP,

$$\begin{cases} y'' + 2y' + y = \delta(t-u), \\ y(0) = 0, \\ y'(0) = 0 \end{cases}$$

There are two ways to make sense of this. The first way is to follow the derivation from Lecture 22. This says that, $y_\delta(t,u) = 0$ for $t < u$, and for $t > u$, it is the solution of the IVP,

$$\begin{cases} y'' + 2y' + y = 0, \\ y(u) = 0, \\ y'(u) = 1 \end{cases}$$

The characteristic polynomial is $z^2 + 2z + 1 = (z+1)^2$. Therefore the general solution is $e^{-(t-u)}, (t-u)e^{-(t-u)}$, i.e.,

$$y(t) = Ae^{-(t-u)} + B(t-u)e^{-(t-u)},$$

for some choice of the constants A and B. The initial conditions give the linear relations,

$$\begin{cases} 1 \cdot A & + & 0 \cdot B & = & 0, \\ -1 \cdot A & + & 1 \cdot B & = & 1, \end{cases}$$

In other words, $A = 0$ an $B = 1$. So the solution is,

$$y_\delta(t,u) = \begin{cases} (t-u)e^{-(t-u)}, & t \geq u, \\ 0, & t < u \end{cases}$$

The other method is to apply the Laplace operator. Denote $Y_u(s) = \mathcal{L}[y_\delta(t,u)]$. Then $\mathcal{L}[y'_\delta] = sY_u(s)$ and $\mathcal{L}[y''_\delta] = s^2Y_u(s)$. On the other hand, $\mathcal{L}[\delta(t-u)] = e^{-su}$. This gives the relation,

$$(s^2 + 2s + 1)Y_u(s) = e^{-su}, \quad Y_u(s) = \frac{1}{(s+1)^2}e^{-su}.$$

The inverse Laplace transform of $1/(s+1)^2$ is $k(t) = te^{-t}$. Therefore by Item (28) from the Laplace transform table in the inside back cover,

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}e^{-su}\right] = S(t-u)k(t-u) = S(t-u)(t-u)e^{-(t-u)}.$$

This agrees with the computation above.

(5)(10 points) p. 335, Problem 12

Solution: By the last exercise, for this ODE the function $y_\delta(t,u)$ is $k(t-u)$, where $k(t) = S(t)te^{-t}$. Therefore by Theorem 3, the solution of the IVP is,

$$y(t) = k(t) * (S(t)\sin(t)) = \int_0^t ue{-}u\sin(t-u)dt = \sin(t)\int_0^t ue^{-u}\cos(u)du - \cos(t)\int_0^t ue^{-u}\sin(u)du.$$

This integral can be computed by several applications of integration by parts. A faster (though much less straightforward) method is to introduce a dummy variable s and compute the function,

$$F(s) = \int_0^t e^{-su}e^{iu}du = \int_0^t e^{-(s-i)u} = -\frac{1}{(s-i)}\left(e^{-(s-i)u}\right|_0^t = \frac{1}{(s-i)}(1 - e^{-(s-i)t}).$$

On the one hand, the derivative with respect to s is,

$$F'(s) = \int_0^t -ue^{-su}e^{iu}du,$$

so that we have,

$$-\mathrm{Re}(F'(1)) = \int_0^t ue^{-su}\cos(u)du, \quad -\mathrm{Im}(F'(1)) = \int_0^t ue^{-su}\sin(u)du.$$

On the other hand, the derivative is,

$$F'(s) = \frac{1}{(s-i)^2}[te^{-(s-i)t}(s-i) - (1 - e^{-(s-i)t})].$$

Therefore,

$$F'(1) = \frac{1}{1-i}te^{-t}(\cos(t) + i\sin(t)) - \frac{1}{(1-i)^2}[(1 - e^{-t}\cos(t)) - ie^{-t}\sin(t)].$$

Rationalizing denominators, this is,

$$F'(1) = \frac{1+i}{2}te^{-t}(\cos(t) + i\sin(t)) - \frac{i}{2}[(1 - e^{-t}\cos(t)) - ie^{-t}\sin(t)],$$

i.e.,

$$-F'(1) = \frac{1}{2}\left[e^{-t}\sin(t) + te^{-t}(\sin(t) - \cos(t))\right] + \frac{i}{2}\left[1 - e^{-t}\cos(t) - te^{-t}(\sin(t) + \cos(t))\right].$$

Therefore,

$$\int_0^t ue^{-u}\cos(u)du = \frac{1}{2}\left[e^{-t}\sin(t) + te^{-t}(\sin(t) - \cos(t))\right],$$

and,

$$\int_0^t ue^{-u}\sin(u)du = \frac{1}{2}\left[1 - e^{-t}\cos(t) - te^{-t}(\sin(t) + \cos(t))\right].$$

Substituting this into the equation for $y(t)$ and simplifying gives the equation,

$$y(t) = \frac{1}{2}\left[-\cos(t) + e^{-t} + te^{-t}\right].$$

Of course this is correct, but it is much less direct than simply computing the Laplace transform. A bit more generally, let $y(t)$ be the solution of the IVP,

$$\begin{cases} y'' + 2y' + y = \sin(t), \\ y(0) = y_0, \\ y'(0) = v_0 \end{cases}$$

Denote $Y(s) = \mathcal{L}[y(t)]$. Then $\mathcal{L}[y'(t)] = sY(s) - y_0$, and $\mathcal{L}[y''(t)] = s^2Y(s) - v_0 - sy_0$. Also, $\mathcal{L}[\sin(t)] = 1/(s^2+1)$. This gives the equation,

$$(s^2+2s+1)Y(s) - (y_0 s + (v_0 + 2y_0)) = \frac{1}{s^2+1}, \quad (s^2+2s+1)Y(s) = y_0(s+1) + (v_0+y_0) + \frac{1}{s^2+1}.$$

Simplifying gives,

$$Y(s) = y_0\frac{1}{s+1} + (v_0+y_0)\frac{1}{(s+1)^2} + \frac{s}{(s+1)^2(s^2+1)}.$$

By the Heaviside coverup method, the partial fraction expansion of the last term is,

$$\frac{1}{(s+1)^2(s^2+1)} = \frac{1}{2}\frac{1}{(s+1)^2} + A\frac{1}{s+1} + B\frac{1}{s^2+1} + C\frac{s}{s^2+1}.$$

Plugging in $s=0$ gives the identity,

$$1 = \frac{1}{2} + A + B, \quad A + B = \frac{1}{2}.$$

Plugging in $s=1$ gives the identity,

$$\frac{1}{8} = \frac{1}{8} + \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C, \quad C = -(A+B) = -\frac{1}{2}.$$

Finally, plugging in $s=2$ gives the identity,

$$\frac{1}{45} = \frac{1}{18} + \frac{1}{3}A + \frac{1}{5}B - \frac{1}{5}.$$

Plugging in $B = \frac{1}{2} - A$ and solving gives the equation $A = \frac{1}{2}$. Thus $B = 0$. Therefore the partial fraction expansion is,

$$\frac{1}{(s+1)^2(s^2+1)} = \frac{1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{s}{s^2+1}.$$

Therefore,

$$Y(s) = y_0\frac{1}{s+1} + (v_0+y_0)\frac{1}{(s+1)^2} + \left[\frac{1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{s}{s^2+1}\right].$$

Therefore,

$$y(t) = \mathcal{L}^{-1}[Y(s)] = y_0e^{-t} + (v_0+y_0)te^{-t} + \frac{1}{2}\left[te^{-t} + e^{-t} - \cos(t)\right].$$

In particular, if $y_0 = v_0 = 0$, this agrees with the computation of the convolution above. (**Moral:** Whenever possible, use the Laplace transform to avoid explicitly computing the convolution).

(6)(10 points) p. 335, Problem 25

Solution: Of course any such function really should be interpreted as "do some manipulation which is rigorous if δ is a usual function, and which leads to the formal expression given". In this vein, let $f(t)$ be a locally bounded, piecewise continuous function on $(-\infty,\infty)$, and let $g(t)$ be a continuous function of compact support. Let $a > 0$ be a real number. Then, using the change of variables $u = at, du = adt$, we have the equation,

$$\int_{-\infty}^{\infty} f(at)g(t)dt = \int_{-\infty}^{\infty} f(u)g(u/a)\frac{1}{a}du = \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)g(u/a)du.$$

Let $a < 0$ be a real number. Then, using the change of variables $u = at, du = adt$, we have the equation,

$$\int_{-\infty}^{\infty} f(at)g(t)dt = \int_{\infty}^{-\infty} f(u)g(u/a)\frac{1}{a}du = \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)g(u/a)du.$$

Now applying the two identities above when $f(t) = \delta(t)$ leads to the formal expressions,

$$\int_{-\infty}^{\infty} \delta(at)g(t)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(u)g(u/a)du = \frac{1}{|a|}g(0) = \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(t)g(t)dt.$$

Therefore, as generalized functions, $\delta(at) = (1/|a|)\delta(t)$. (This can be made completely rigorous by expressing $\delta(t)$ as a limit of honest functions f_h as $h \to 0$ and the applying the equations above to f_h. However, this requires a topology on the space of generalized functions, which I don't want to describe.)

Another derivation involves the Laplace transform. First of all, clearly $\delta(t)$ is an even function. So for $a < 0$, $\delta(at) = \delta(|a|t)$. So it suffices to consider the case that $a > 0$.

For any function $y(t)$ of exponential order, denoting $Y(s) = \mathcal{L}[y(t)]$, we have the identity,

$$\mathcal{L}[y(at)] = \frac{1}{a}Y(s/a).$$

This was one of the identities from lecture. For the Dirac delta function, $\mathcal{L}[\delta(t)] = 1$. Therefore,

$$\mathcal{L}[\delta(at)] = \frac{1}{a}1 = \mathcal{L}\left[\frac{1}{a}\delta(t)\right].$$

This again gives the identity, $\delta(at) = (1/|a|)\delta(t)$.