18.034 SOLUTIONS TO PROBLEM SET 5

Due date: Friday, April 2 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

(1)(10 points) Read *Spotlight on Approximating Functions* on pp 616–630. Everybody will receive full credit for this problem. If you have any questions about the material, want to discuss the material further, etc., please come and talk with me or Edward during office hours. There is a serious mistake in the proof of Theorem 9, p. 626 (which is easily corrected by using Theorem 13). Can you see what the mistake is?

Remark: This was not to be turned in. There is a typo in Theorem 9 – all instances of $\mathbf{C}^0$ should be $\mathbf{C}^1$. A more serious mistake is that it is only proved that the Fourier series of f converges uniformly to a continuous function, but it is not proved that this continuous function is f. In Theorem 13 it is proved that the Fourier series converges pointwise to f, which finishes the proof that the Fourier series converges uniformly to f.

(2)(10 points) The Riemann zeta function $\zeta(s)$ is defined for real numbers $s > 1$ by the formula,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(In fact the Riemann zeta function can be defined as an *analytic function* for every complex number s except 1, but certainly not by the series above.) The values of $\zeta(s)$ are of importance throughout mathematics, and one of the most famous open problems in mathematics is to prove that every root of $\zeta(s)$ is of the form $\frac{1}{2} + bi$ for some real number b.

On the interval $[-\pi, \pi]$, consider the orthonormal sequence $\Phi = (\phi_n)_{n\in\mathbb{Z}}$ where,

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.$$

Let u be a real number and consider the function $f_u(x) = e^{ux}$.

(a)(5 points) Compute the Fourier coefficients,

$$\langle f_u, \phi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ux} e^{-inx} dx.$$

Solution: If $u = 0$, then $\langle f_0, \phi_0 \rangle = \sqrt{2\pi}$ and $\langle f_0, \phi_n \rangle = 0$ for $n \neq 0$. Thus suppose that $u \neq 0$. Then for every n,

$$\langle f_u, \phi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{(u-in)x} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{(u-in)} \left(e^{(u-in)x} \Big|_{-\pi}^{\pi} \right.$$

Of course $e^{-in\pi} = e^{in\pi} = (-1)^n$. So the Fourier coefficient is,

$$\langle f_u, \phi_n \rangle = \frac{(-1)^n}{\sqrt{2\pi}} \frac{e^{\pi u} - e^{-\pi u}}{u - in}.$$

(b)(5 points) Apply Plancherel's theorem,

$$\langle f_u, f_u \rangle = \sum_{n\in\mathbb{Z}} |\langle f_u, \phi_n \rangle|^2,$$

to get an equation that can be used to find a formula,

$$\sum_{n=1}^{\infty} \frac{1}{u^2+n^2} = g(u),$$

where $g(u)$ is some simple expression involving exponentials, etc. (**Hint**: One formulation of the answer involves the hyperbolic cotangent).

Solution: For $u = 0$, there is only one nonzero Fourier coefficient, and Plancherel's theorem simply gives,

$$\int_{-\pi}^{\pi} 1^2 dx = |\sqrt{2\pi}|^2 = 2\pi,$$

which is clearly true. The interesting case is if $u \neq 0$. Then we have,

$$|\langle f_u, \phi_n \rangle|^2 = \frac{1}{2\pi} \frac{(e^{\pi u} - e^{-\pi u})^2}{|u - in|^2}.$$

Remember that $u - in$ is a complex number. Therefore $|u - in|^2 = u^2 + n^2$. Therefore,

$$|\langle f_u, \phi_n \rangle|^2 = \frac{1}{2\pi} \frac{(e^{\pi u} - e^{-\pi u})^2}{u^2 + n^2}.$$

Applying Plancherel's theorem,

$$\int_{-\pi}^{\pi} |e^{ux}|^2 dx = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \frac{(e^{\pi u} - e^{-\pi u})^2}{u^2+n^2}.$$

First of all, $|e^{ux}|^2$ is just e^{2ux}. So the integral is,

$$\int_{-\pi}^{\pi} e^{2ux} dx = \frac{e^{2\pi u} - e^{-2\pi u}}{2u}.$$

As for the sum, factoring common terms gives,

$$\frac{(e^{\pi u} - e^{-\pi u})^2}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{u^2+n^2}.$$

Notice that the term for $-n$ is the same as the term for n. Singling out the term for $n = 0$, the series reduces to,

$$\frac{(e^{\pi u} - e^{-\pi u})^2}{2\pi} \left(\frac{1}{u^2} + 2 \sum_{n=1}^{\infty} \frac{1}{u^2+n^2} \right).$$

So we have the equation,

$$\frac{e^{2\pi u} - e^{-2\pi u}}{2u} = \frac{(e^{\pi u} - e^{-\pi u})^2}{2\pi} \left(\frac{1}{u^2} + 2 \sum_{n=1}^{\infty} \frac{1}{u^2+n^2} \right).$$

Dividing both sides by $(e^{\pi u} - e^{-\pi u})^2/2\pi$ gives,

$$\frac{\pi}{u} \frac{e^{2\pi u} - e^{-2\pi u}}{(e^{\pi u} - e^{-\pi u})^2} = \frac{1}{u^2} + 2 \sum_{n=1}^{\infty} \frac{1}{u^2+n^2}.$$

The term $(e^{2\pi u} - e^{-2\pi u})/(e^{\pi u} - e^{-\pi u})^2$ is the same as $(v^2 - w^2)/(v - w)^2$ where $v = e^{\pi u}$ and $w = e^{-\pi u}$. Since the numerator is a difference of squares, this simplifies to $(v + w)/(v - w)$. Making this simplification and solving for the series gives,

$$\sum_{n=1}^{\infty} \frac{1}{u^2+n^2} = \frac{\pi}{2u} \frac{e^{\pi u} + e^{-\pi u}}{e^{\pi u} - e^{-\pi u}} - \frac{1}{2u^2}.$$

Recall that the hyperbolic cotangent is defined by,

$$\coth(y) = \frac{e^y + e^{-y}}{e^y - e^{-y}}.$$

Therefore the series is,

$$\sum_{n=1}^{\infty} \frac{1}{u^2 + n^2} = \frac{\pi}{2u} \coth(\pi u) - \frac{1}{2u^2}.$$

It is worth remarking that the series above converges uniformly for all u. Indeed the tail of the series is less than or equal to the tail of the series for $u = 0$, which is the tail of the convergent series $\sum_{n=1}^{\infty} 1/n^2$. The right-hand-side, however, is only defined for $u \neq 0$. However, the singularity at $u = 0$ is a removable singularity. The continuous extension of the right-hand-side is a function that is infinitely differentiable for all u.

(c)(5 points extra credit) Consider the Taylor expansion of both sides of the equation above. Use this to find an expression, for each even integer $2n > 0$, of the value $\zeta(2n)$ in terms of a sequence of numbers that satisfies a recursion relation you can write down. Compute the values of $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$. (**Remark**: Of course there is great freedom in the sequence of numbers to use. The most common choice is to express $\zeta(2n)$ in terms of the Bernoulli numbers, B_n, which are defined by the following,

$$\frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n-1}.$$

The recursion relation for B_n is easy to write down.)

Solution: The basic idea is to compute the Taylor expansion of the continuous extension of the right-hand-side of the equation above. The basic Taylor expansion is,

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

where 0! is defined to be 1. When we sum $e^z + e^{-z}$, all the terms involving an odd power of z cancel. When we sum $e^z - e^{-z}$, all the terms involving an even power of z cancel. This gives,

$$e^{\pi u} + e^{-\pi u} = 2 \sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n)!} u^{2n},$$

and

$$e^{\pi u} - e^{-\pi u} = 2 \sum_{n=0}^{\infty} \frac{\pi^{2n+1}}{(2n+1)!} u^{2n+1}.$$

We write the second series as,

$$\frac{1}{\pi u}(e^{\pi u} - e^{-\pi u}) = 2 \sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n+1)!} u^{2n}.$$

Now we introduce the power series,

$$\left[e^{\pi u} + e^{-\pi u}\right] / \left[\frac{1}{\pi u}(e^{\pi u} - e^{-\pi u})\right] = \sum_{n=0}^{\infty} \pi^{2n} a_n u^{2n}.$$

Here the coefficients are to be determined. To find the coefficients, we use the identity,

$$2\sum_{n=0}^{\infty}\frac{\pi^{2n}}{(2n)!}u^{2n} = e^{\pi u}+e^{-\pi u} = [\frac{1}{\pi u}(e^{\pi u}-e^{-\pi u})]\sum_{m=0}^{\infty}\pi^{2m}a_m u^{2m}$$

$$= 2\left(\sum_{l=0}^{\infty}\frac{\pi^{2l}}{(2l+1)!}u^{2l}\right)\left(\sum_{m=0}^{\infty}\pi^{2m}a_m u^{2m}\right).$$

Multiplying the two series term-by-term gives,

$$2\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\pi^{2(m+l)}a_m}{(2l+1)!}u^{2(m+l)}.$$

Gathering all terms whose exponent is $2n$ gives,

$$2\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{a_m}{(2n-2m+1)!}\right)\pi^{2n}u^{2n}.$$

Since the two power series are convergent and equal, for each n the coefficient of u^{2n} of each series is equal. This gives,

$$2\frac{1}{(2n)!}\pi^{2n} = 2\sum_{m=0}^{n}\frac{a_m}{(2n-2m+1)!}\pi^{2n}.$$

Of course $2\pi^{2n}$ cancels from each side of the equation. Plugging in $n=0$ gives $a_0=1$. For each $n>0$, solving for a_n gives,

$$a_n = \frac{1}{(2n)!}-\sum_{m=0}^{n-1}\frac{a_m}{(2n-2m+1)!}.$$

This gives a recursive algorithm for finding the numbers a_n. It is interesting to notice that each a_n is a fraction (this can be proved by an easy induction argument).

Plugging in the power series for the right-hand-side of our original equation,

$$\frac{\pi}{2u}(\frac{1}{\pi u}\sum_{n=0}^{\infty}a_n\pi^{2n}u^{2n}) - \frac{1}{2u^2} = \frac{1}{2u^2}((a_0+\sum_{n=1}^{\infty}a_n\pi^{2n}u^{2n})-1).$$

Since $a_0=1$, this cancels and we get,

$$\sum_{n=1}^{\infty}\frac{1}{u^2+n^2} = \sum_{m=0}^{\infty}\frac{\pi^{2m+2}}{2}a_{m+1}u^{2m}.$$

On the other hand, for each integer n, we can expand $1/(u^2+n^2)$ as a geometric series,

$$\frac{1}{u^2+n^2} = \frac{1}{n^2}\frac{1}{1+(u/n)^2} =$$

$$\frac{1}{n^2}\sum_{m=0}^{\infty}(-1)^m\frac{1}{n^{2m}}u^{2m} = \sum_{m=0}^{\infty}(-1)^m\frac{1}{n^{2m+2}}u^{2m}.$$

Plugging this in gives the series,

$$\sum_{n=1}^{\infty}\frac{1}{u^2+n^2} = \sum_{n=1}^{\infty}\left(\sum_{m=0}^{\infty}(-1)^m\frac{1}{n^{2m+2}}u^{2m}\right).$$

We would like to interchange the sums in this last series. This is justified if the series is absolutely convergent, i.e., if the series of absolute values is convergent. Let $|u| < 1$. Then the series of absolute values is,

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n^{2m+2}}|u|^{2m} = \sum_{n=1}^{\infty}\frac{1}{n^2 - |u|^2}.$$

Of course this series is dominated by the series,

$$\frac{1}{1-|u|^2} + \sum_{n=2}^{\infty}\frac{1}{n^2-1}.$$

By the integral test, for example, this last series is convergent. Therefore the original series is absolutely convergent which justifies interchanging the sums. This gives,

$$\sum_{m=0}^{\infty}(-1)^m\left(\sum_{n=1}^{\infty}\frac{1}{n^{2m+2}}\right)u^{2m} = \sum_{m=0}^{\infty}(-1)^m\zeta(2m+2)u^{2m}.$$

Finally, we get the equality of convergent power series,

$$\sum_{m=0}^{\infty}(-1)^m\zeta(2m+2)u^{2m} = \sum_{m=0}^{\infty}\frac{\pi^{2m+2}}{2}a_{m+1}u^{2m}.$$

Therefore the coefficient of u^{2m} on each side of the equation is equal, i.e.,

$$\zeta(2m+2) = (-1)^m\frac{a_{m+1}}{2}\pi^{2m+2},$$

or,

$$\zeta(2m) = (-1)^{m-1}\frac{a_m}{2}\pi^{2m}.$$

This is a true expression, and the recursive formula can be readily used to compute any particular value of $\zeta(2m)$. The standard convention, however, is to use the Bernoulli numbers B_n rather than a_n. The relation between the two of them follows from the straightforward computation,

$$\frac{1}{e^z-1} + \frac{1}{2} - \frac{1}{z} = \frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z} = \sum_{n=1}^{\infty}\frac{a_n}{2^{2n}}z^{2n-1}.$$

This gives the identities,

$$\begin{aligned} B_n &= \tfrac{(-1)^{n-1}(2n)!}{2^{2n}}a_n, \\ a_n &= \tfrac{(-1)^{n-1}2^{2n}}{(2n)!}B_n \end{aligned}$$

In terms of the Bernoulli numbers,

$$\zeta(2m) = \frac{2^{2m-1}}{(2m)!}B_m\pi^{2m}.$$

In particular, each value of $\zeta(2m)$ is a rational number times π^{2m}.

Using the recursion relation, we have $a_0 = 1$, $a_1 = 1/3$, $a_2 = -1/45$ and $a_3 = 2/945$. This gives, $B_1 = 1/6$, $B_2 = 1/30$, and $B_3 = 1/42$. So the first three even zeta values are,

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945}.$$

Many beautiful results involving the Riemann zeta function and Bernoulli numbers can be found at the following URL.

`http://mathworld.wolfram.com/RiemannZetaFunction.html`

Each of the following problems is from the textbook. The point value of the problem is next to the problem.

(3)(5 points) p. 581, Problem 11

Solution: First of all, the normalized Fourier trigonometric functions on the interval $[-\pi, \pi]$ are,

$$\begin{aligned} c_0(x) &= \tfrac{1}{\sqrt{2\pi}}, \\ c_n(x) &= \tfrac{1}{\sqrt{\pi}}\cos(nx), \quad n = 1, 2, \ldots \\ s_n(x) &= \tfrac{1}{\sqrt{\pi}}\sin(nx), \quad n = 1, 2, \ldots \end{aligned}$$

Clearly $f(x)$ is an odd function. Therefore the only nonzero Fourier coefficients are the coefficients of s_n. And these are,

$$\langle f, s_n \rangle = \frac{1}{\sqrt{\pi}} 2 \int_0^B (-A)\sin(nx)dx.$$

This is just,

$$\frac{2A}{\sqrt{\pi}} \left(\frac{1}{n}\cos(nx) \right|_0^B = -\frac{2A}{\sqrt{\pi}} \frac{1-\cos(nB)}{n}.$$

Therefore the Fourier series equals the Fourier sine series, which is,

$$FS(f) = \sum_{n=1}^{\infty} -\frac{A}{\pi}\frac{1-\cos(nB)}{n}\sin(nx).$$

(4)(10 points) p. 581, Problem 23

Solution, (a): The function $f(x) = |x|$ is clearly an even function. Therefore the Fourier coefficients, $\langle f, s_n \rangle$ are all zero. First of all,

$$\langle f, c_0 \rangle = \frac{1}{\sqrt{2\pi}} 2 \int_0^{\pi} x dx = \frac{2}{\sqrt{2\pi}} \left(\frac{x^2}{2} \right|_0^{\pi} = \sqrt{\frac{\pi^3}{2}}.$$

For each $n > 0$,

$$\langle f, c_n \rangle = \frac{1}{\sqrt{\pi}} 2 \int_0^{\pi} x\cos(nx)dx.$$

We solve this by integration by parts, $u = x, dv = \cos(nx)$ and $du = dx, v = 1/n\sin(nx)$. This gives,

$$\frac{2}{\sqrt{2\pi}} \left(\frac{1}{n} x\sin(nx) \right|_0^{\pi} + \frac{2}{\sqrt{2\pi}} \int_0^{\pi} -\frac{1}{n}\sin(nx)dx.$$

The first term is zero. The second term is easy to integrate. So,

$$\langle f, c_n \rangle = \frac{2}{\sqrt{2\pi}} \left(\frac{1}{n^2}\cos(nx) \right|_0^{\pi} = \frac{2}{\sqrt{2\pi}} \frac{1}{n^2} \left((-1)^n - 1 \right).$$

This is zero if n is even. And for $n = 2k + 1$ an odd integer, it is,

$$\langle f, c_{2k+1} \rangle = -\frac{4}{\sqrt{2\pi}} \frac{1}{(2k+1)^2}.$$

Therefore the Fourier series is,

$$FS(f) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos((2k+1)x).$$

In particular, plugging in $x = 0$, gives,

$$FS(f)(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

Since the periodic extension $\widetilde{f}$ is continuous and piecewise smooth, Theorem 10.2.2 implies that the Fourier series converges pointwise to $\widetilde{f}$ at every point of $\mathbb{R}$. In particular, for $x = 0$, the Fourier series converges pointwise to $f(0) = 0$, i.e.,

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

Therefore,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

(b) As mentioned above, the periodic extension $\widetilde{f}$ of $|x|$ is continuous and piecewise smooth. Therefore, by Theorem 10.2.2, the Fourier series $FS[f](x)$ converges pointwise to $\widetilde{f}(x)$ for every x in $\mathbb{R}$.

(5)(5 points) p. 588, Problem 7

Solution: On the interval $[-\pi, \pi]$ the normalized Fourier exponential functions are,

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}.$$

These functions are orthonormal. In this case, $f(x) = \sqrt{2\pi}\phi_1(x)$. Therefore $\langle f, \phi_n\rangle = \sqrt{2\pi}\langle\phi_1, \phi_n\rangle$. This is 0 unless $n = 1$, and it is $\sqrt{2\pi}$ for $n = 1$. So the Fourier exponential series is simply,

$$e^{ix} = \sqrt{2\pi}\phi_1(x) = e^{ix}.$$

(Not a very exciting Fourier series!)

(6)(10 points) p. 588, Problem 17

Solution: The solution is more readable if we use variable names for the constants. Denote by m the mass of the spring, $m = 1$ kg. Denote by k the spring constant, $k = 1.01\ \frac{\text{N}}{\text{m}}$. Denote by b the damping constant, $b = 0.2\ \frac{\text{N}}{\text{s}^2}$. And denote by F_0 the amplitude of the square wave driving force, i.e., $F_0 = 1$ N.

Denote by $x(t)$ the position of the spring at time t. The differential equation describing the motion is,

$$mx''(t) + bx'(t) + kx(t) = f(t).$$

Because $f(t)$ is periodic of period 2π, in the steady state $x(t)$ will be a twice differentiable, periodic function of period 2π. Therefore the Fourier exponential series of $x(t)$ will converge to $x(t)$ (with respect to the interval $[-\pi, \pi]$).

The normalized Fourier exponential functions on $[-\pi, \pi]$ are $\phi_n(t) = \frac{1}{\sqrt{2\pi}}e^{inx}$. Define $p(D)$ to be,

$$p(D) = mD^2 + bD + k.$$

Then the Fourier coefficients $a_n(x) = \langle x, \phi_n\rangle$ satisfy the equation,

$$p(in)a_n(x) = a_n(f), \quad p(in) = -(mn^2 - k) + ibn.$$

Because $b \neq 0$, $p(in)$ is nonzero for each n. Therefore we can solve to get,

$$a_n(x) = \frac{1}{p(in)} a_n(f) = q(n) a_n(f),$$

where $q(n)$ is the function,

$$q(n) = -\frac{1}{(mn^2 - k)^2 + b^2 n^2} \left((mn^2 - k) + ibn \right).$$

It remains to compute the Fourier coefficients $a_n(f)$.

For $n = 0$,

$$a_0(f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} F_0 dt = F_0 \sqrt{\frac{\pi}{2}}.$$

Let n be different than 0. Then,

$$a_n(f) = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} F_0 e^{-inx} dx = \frac{iF_0}{n\sqrt{2\pi}} \left(e^{-inx} \right|_0^{\pi} = \frac{iF_0}{n\sqrt{2\pi}} \left((-1)^n - 1 \right).$$

Therefore $a_n(f) = 0$ if n is even. If n is odd, then

$$a_n(f) = -\frac{i2F_0}{n\sqrt{2\pi}}.$$

Substituting this into our equation for $a_n(x)$ gives,

$$a_0(x) = \frac{F_0}{k} \sqrt{\frac{\pi}{2}},$$

for n even,

$$a_n(x) = 0,$$

and for n odd,

$$a_n(x) = q(n) a_n(f) = \frac{2F_0}{n\sqrt{2\pi}} \frac{1}{(mn^2 - k)^2 + b^2 n^2} \left(-bn + i(mn^2 - k) \right).$$

We can simplify this somewhat by defining,

$$A_n = \frac{\sqrt{2\pi}}{2} |a_n(x)| = \frac{F_0}{n} \frac{1}{\sqrt{(mn^2 - k)^2 + b^2 n^2}},$$

and defining,

$$\phi_n = \tan^{-1}(\text{Im}(a_n(x))/\text{Re}(a_n(x))) = \tan^{-1} \left(\frac{mn^2 - k}{bn} \right).$$

Then we have,

$$a_n(x) = \frac{2}{\sqrt{2\pi}} A_n e^{-i\phi_n}.$$

Of course $A_{-n} = A_n$, and $\phi_{-n} = -\phi_n$. Therefore the Fourier series is,

$$FS[x] = a_0(f) \frac{1}{\sqrt{2\pi}} + \sum_{l=0}^{\infty} \frac{1}{\sqrt{2\pi}} \left(a_{2l+1}(x) e^{i(2l+1)t} + a_{-(2l+1)} e^{-i(2l+1)t} \right)$$

$$= a_0(f) \frac{1}{\sqrt{2\pi}} + \sum_{l=0}^{\infty} \frac{A_{2l+1}}{2} \left(\exp[i((2l+1)t - \phi_{2l+1})] + \exp[-i((2l+1)t - \phi_{2l+1})] \right).$$

For any real number θ, $\exp(i\theta)+\exp(-i\theta)$ equals $2\cos(\theta)$. Therefore the Fourier series for x reduces to,

$$FS[x] = a_0(f)\frac{1}{\sqrt{2\pi}} + \sum_{l=0}^{\infty} A_{2l+1}\cos((2l+1)t - \phi_{2l+1}).$$

Plugging in for $a_0(f)$ and A_{2l+1}, and changing the name of the dummy variable l to n,

$$FS[x] = \frac{F_0}{2k} + \sum_{n=0}^{\infty} \frac{F_0}{2n+1}\frac{1}{\sqrt{(m(2n+1)^2-k)^2 + b^2(2n+1)^2}}\cos((2n+1)t - \phi_{2n+1}).$$

Let's do a little more analysis (although this is *not* required for this problem). Define $\omega_n = (2n+1)$, the frequency of the n^{th} term in the Fourier sequence. Define $u_n = \omega_n^2$. Then the n^{th} term in the Fourier sequence is,

$$F_0\frac{1}{\omega_n\sqrt{(m\omega_n^2-k)^2 + b^2\omega_n^2}}\cos(\omega_n t - \phi_{2n+1}).$$

In particular, the square of the amplitude is,

$$F_0^2\frac{1}{u_n\left((mu_n-k)^2 + b^2u_n\right)}.$$

The *near resonant term* is the term such that the square of the amplitude is largest, i.e., such that the following expression is smallest,

$$u_n\left((mu_n-k)^2 + b^2u_n\right).$$

Consider the function,

$$g(u) = u\left((mu-k)^2 + b^2u\right).$$

This is a cubic polynomial in u, it is negative for $u < 0$, it is positive for $u > 0$, and it has precisely one root at $u = 0$. There will be 2 critical points of this function, one of which is a local minimum and one of which is a local maximum. They occur when,

$$g'(u) = \left((mu-k)^2 + b^2u\right) + u\left(2m(mu-k) + b^2\right),$$

is zero. By the quadratic equation, the local minimum is,

$$u_c = \frac{2mk - b^2}{3m^2} + \frac{1}{3m^2}\sqrt{b^4 - 4mkb^2 + m^2k^2}.$$

The near resonant term will be either $n = 0$ or the value of n such that u_n is nearest to u_c.

For the constants in our particular problem,

$$u_c = 0.969138588\ldots$$

So the near resonant term will be the one for which u_n is closest to 1, i.e. $n = 0$. The near resonant term is,

$$F_0\frac{1}{\sqrt{(m-k)^2 + b^2}}\cos(t - \phi_{2n+1}) \approx 4.994\cos(t - 0.05).$$