18.034 PROBLEM SET 4

Due date: Friday, March 12 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Each of the following problems is from the textbook. The point value of the problem is next to the problem.

(1)(5 points) p. 184, Problem 24

Solution: The characteristic polynomial is,

$$z^2 + 2z + 65.$$

The discriminant of this quadratic polynomial is,

$$2^2 - 4 \times 65 = -4 \times 64 = -(16)^2.$$

Therefore the roots are the complex conjugates,

$$\lambda_\pm = -1 \pm 8i.$$

The general complex solution of the ODE is,

$$\widetilde{y}(t) = \widetilde{C}_+ e^{\lambda_+ t} + \widetilde{C}_- e^{\lambda_- t}.$$

The general real solution of the ODE is,

$$y(t) = C_1 e^{-t}\cos(8t) + C_2 e^{-t}\sin(8t).$$

The derivative of $y(t)$ is given by,

$$y'(t) = C_1(-e^{-t}\cos(8t) - 8e^{-t}\sin(8t)) + C_2(-e^{-t}\sin(8t) + 8e^{-t}\cos(8t)).$$

Therefore the IVP,

$$\begin{cases} y'' + 2y' + 65 = 0, \\ y(0) = y_0, \\ y'(0) = v_0 \end{cases}$$

leads to the system of 2 linear equations in 2 unknowns,

$$\begin{cases} C_1 + 0C_2 = y_0, \\ -C_1 + 8C_2 = v_0 \end{cases}$$

The solution of this system is,

$$\begin{cases} C_1 = y_0 \\ C_2 = \frac{1}{8}y_0 + \frac{1}{8}v_0 \end{cases}$$

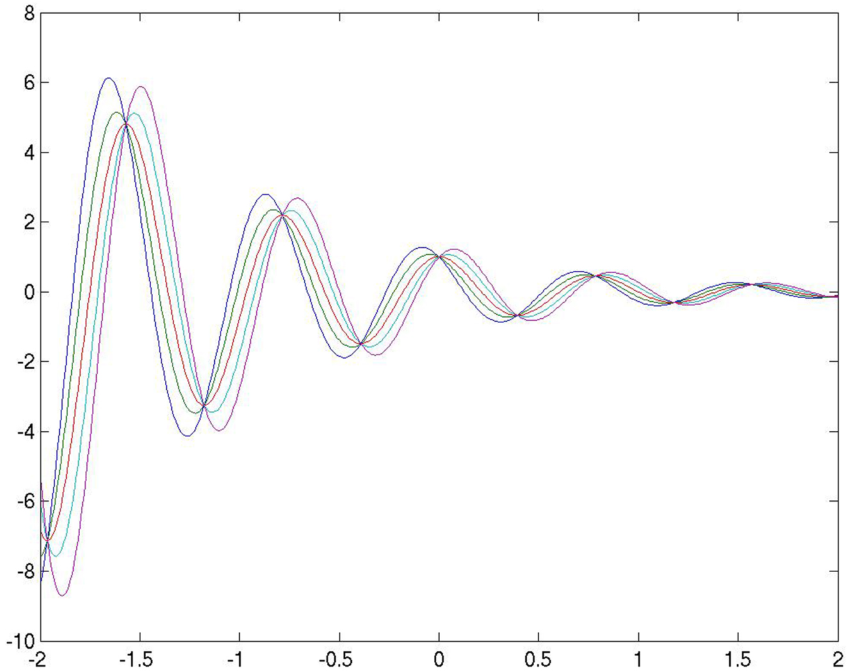
In other words,

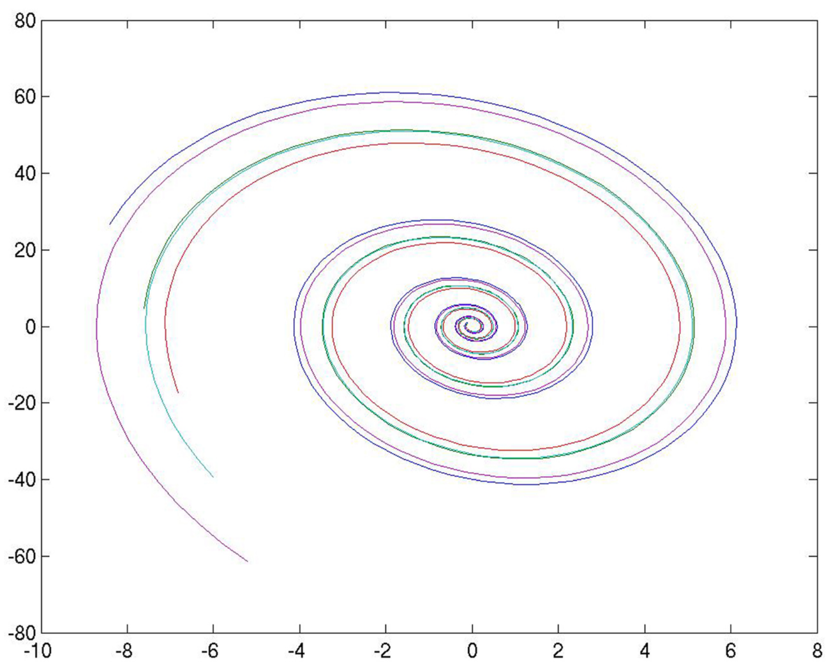
$$y(t) = \frac{1}{8}y_0 e^{-t}(8\cos(8t) + \sin(8t)) + \frac{1}{8}v_0 e^{-t}\sin(8t).$$

Therefore the 5 solution curves are,

$$\begin{cases} y_{(1,-6)}(t) = \frac{1}{8}e^{-t}(8\cos(8t) + \sin(8t)) - \frac{6}{8}e^{-t}\sin(8t), \\ y_{(1,-3)}(t) = \frac{1}{8}e^{-t}(8\cos(8t) + \sin(8t)) - \frac{3}{8}e^{-t}\sin(8t), \\ y_{(1,0)}(t) = \frac{1}{8}e^{-t}(8\cos(8t) + \sin(8t)), \\ y_{(1,+3)}(t) = \frac{1}{8}e^{-t}(8\cos(8t) + \sin(8t)) + \frac{3}{8}e^{-t}\sin(8t), \\ y_{(1,+6)}(t) = \frac{1}{8}e^{-t}(8\cos(8t) + \sin(8t)) + \frac{6}{8}e^{-t}\sin(8t) \end{cases}$$

The (t, y)-graphs and the (y, y')-orbits are posted here.





(2)(5 points) p. 185, Problem 32

Solution: By definition of the derivative of a complex-valued function,

$$\begin{aligned} y'(t) &= u'(t) + iv'(t), \\ y''(t) &= u''(t) + iv''(t) \end{aligned}$$

Since a and b are real numbers, the expression $y'' + ay' + by$ equals,

$$\begin{aligned} &(u''(t) + iv''(t)) + (au'(t) + iav'(t)) + (bu(t) + ibv(t)) = \\ &(u''(t) + au'(t) + bu(t)) + i(v''(t) + av'(t) + bv(t)). \end{aligned}$$

A complex-valued function equals 0 iff both the real part and the imaginary part equal 0. Therefore $y(t)$ is a solution of the ODE iff both $u(t)$ and $v(t)$ are solutions of the ODE. In particular, $e^{(\alpha+i\beta)t}$ is a solution of the ODE iff both $u(t) = e^{\alpha t}\cos(\beta t)$ and $v(t) = e^{\alpha t}\sin(\beta t)$ are solutions of the ODE.

If $e^{(\alpha+i\beta)t}$ is a solution of the ODE, then $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are solutions of the ODE. By linearity, also

$$e^{(\alpha-i\beta)t} = 1 \times (e^{\alpha t}\cos(\beta t)) + (-i)(e^{\alpha t}\sin(\beta t)),$$

is a solution of the ODE.

(3)(5 points) p. 206, Problem 22

Solution: The characteristic polynomial equals,

$$z^2 - 3z + 2 = (z-2)(z-1).$$

Therefore the general solution of the homogeneous ODE is,

$$y_h(t) = C_1e^t + C_2e^{2t}.$$

Also, neither 0 nor -1 are roots of the characteristic polynomial. Therefore we are in "Case I" of the method of undetermined coefficients.

First consider a particular solution of the ODE,

$$y_a'' - 3y_a' + 2y_a = 8t^2.$$

By the method of undetermined coefficients, we guess that the solution is a quadratic polynomial with undetermined coefficients, i.e.,

$$y_a(t) = a_2t^2 + a_1t + a_0.$$

This leads to the table,

$2 \times y_a =$	$2 \times($	a_2t^2	$+$	a_1t	$+$	a_0	$)$
$-3 \times y_a' =$	$-3 \times($	$0t^2$	$+$	$2a_2t$	$+$	a_1	$)$
$1 \times y_a'' =$	$1 \times($	$0t^2$	$+$	$0t$	$+$	$2a_2$	$)$
		$2a_2t^2$	$+$	$(-6a_2 + 2a_1)t$	$+$	$(2a_2 - 3a_1 + 2a_0)$	

The solution is,

$$\begin{cases} a_2 = 4 \\ a_1 = 12 \\ a_0 = 14 \end{cases}$$

In other words,

$$y_a(t) = 4t^2 + 12t + 14.$$

Next consider a particular solution of the ODE,

$$y_b'' - 3y_b' + 2y_b = 12e^{-t}.$$

By the method of undetermined coefficients, we guess that the solution is of the form,

$$y_b(t) = b_0 e^{-t}.$$

Substituting this in gives,

$$b_0(-1)^2 e^{-t} - 3b_0(-1)e^{-t} + 2b_0 e^{-t} = 12e^{-t},$$

i.e. $6b_0 = 12$. The solution is $b_0 = 2$, i.e.,

$$y_b(t) = 2e^{-t}.$$

So the general solution of the inhomogeneous ODE is,

$$y_g(t) = (4t^2 + 12t + 14) + (2e^{-t}) + C_1 e^t + C_2 e^{2t}.$$

The derivative is,

$$y_g'(t) = (8t + 12) + (-2e^{-t}) + C_1 e^t + 2C_2 e^{2t}.$$

Plugging in the initial conditions leads to the 2 linear equations in 2 unknowns,

$$\begin{cases} C_1 & + & C_2 & = & 0 - (14) - (2) & = & -16 \\ C_1 & + & 2C_2 & = & 2 - (12) - (-2) & = & -8 \end{cases}$$

By elimination, the solution is,

$$\begin{cases} C_1 & = & -24 \\ C_2 & = & 8 \end{cases}$$

So the solution of the IVP is,

$$y_g(t) = (4t^2 + 12t + 14) + 2e^{-t} - 24e^t + 8e^{2t}.$$

(4)(5 points) p. 211, Problem 6

Solution: The characteristic polynomial is,

$$p(z) = z^2 + 2z + 1 = (z+1)^2.$$

Therefore there is 1 real repeated root $z = -1$. The general solution of the homogeneous ODE is,

$$y_g(t) = C_0 e^{-t} + C_1 t e^{-t}.$$

The differential equation is $p(D)y = e^{-t}$. We guess that the particular solution is of the form $y_d(t) = e^{-t} g(t)$ where $g(t)$ is a polynomial. By the *exponential shift rule*,

$$p(D)[e^{-t} g(t)] = e^{-t} p(D-1) g(t) = e^{-t} D^2 g(t).$$

Thus $y_d(t)$ is a solution iff $D^2 g(t) = 1$. We guess that $g(t) = a_2 t^2$ for some constant a_2 and plug in to get $2a_2 = 1$. The solution is $a_2 = \frac{1}{2}$. Therefore the general real-valued solution of the ODE is,

$$y_g(t) = C_0 e^{-t} + C_1 t e^{-t} + \frac{1}{2} t^2 e^{-t}.$$

(5)(5 points) p. 212, Problem 26

Solution: The characteristic polynomial is,

$$p(z) = z^2 + 2z + 2 = (z+1)^2 + 1.$$

The roots are the complex conjugates,

$$\lambda_{\pm} = -1 \pm i.$$

Therefore the general complex-valued solution of the homogeneous ODE is,

$$\widetilde{y}(t) = \widetilde{C}_+ e^{\lambda_+ t} + \widetilde{C}_- e^{\lambda_- t}.$$

Equivalently, the general complex-valued solution of the homogeneous ODE is,

$$\widetilde{y}(t) = \widetilde{B}_+ e^{\lambda_+ (t-1)} + \widetilde{B}_- e^{\lambda_- (t-1)},$$

just by defining $\widetilde{B}_\pm = \widetilde{C}_\pm e^{\lambda_\pm}$. Therefore the general real-valued solution of the homogeneous ODE is,

$$y_h(t) = B_1 e^{-(t-1)} \cos(t-1) + B_2 e^{-(t-1)} \sin(t-1).$$

Since 0 is not a root of the characteristic polynomial, we guess that the particular solution is a linear polynomial with undetermined coefficients, i.e.,

$$y_a(t) = a_1 t + a_0.$$

This leads to the table,

$2 \times y_a$	$=$	2	$\times($	$a_1 t$	$+$	a_0	$)$
$2 \times y_a'$	$=$	2	$\times($	$0t$	$+$	a_1	$)$
$1 \times y_a''$	$=$	1	$\times($	$0t$	$+$	0	$)$
				$2a_1 t$	$+$	$(2a_1 + 2a_0)$	

The solution is $a_1 = 1$, $a_0 = -1$, i.e.,

$$y_a(t) = t - 1.$$

Therefore the general solution of the inhomogeneous ODE is,

$$y_g(t) = (t-1) + B_1 e^{-(t-1)} \cos(t-1) + B_2 e^{-(t-1)} \sin(t-1).$$

The derivative of $y_g(t)$ is given by,

$$y_g'(t) = 1 + B_1(-e^{-(t-1)} \cos(t-1) - e^{-(t-1)} \sin(t-1)) + B_2(-e^{-(t-1)} \sin(t-1) + e^{-(t-1)} \cos(t-1)).$$

Therefore the IVP,

$$\begin{cases} y'' + 2y' + 2y = 2t, \\ y(1) = 1, \\ y'(1) = 0 \end{cases}$$

leads to the system of 2 linear equations in 2 unknowns,

$$\begin{cases} B_1 + 0B_2 = 1 - (0) = 1 \\ -B_1 + B_2 = 0 - (1) = -1 \end{cases}$$

The solution of this system is,

$$\begin{cases} B_1 = 1 \\ B_2 = 0 \end{cases}$$

In other words, the solution of the IVP is,

$$y(t) = (t-1) + e^{-(t-1)} \cos(t-1).$$

(6)(5 points) p. 221, Problem 6

Solution: Of course the interval of definition of this ODE is $(0, +\infty)$. Let λ be a complex number and consider the function,

$$y(t) = t^\lambda := e^{\lambda \ln(t)}.$$

The derivative and second derivative are,

$$\begin{array}{rclcl} y'(t) & = & e^{\lambda\ln(t)}\lambda\frac{1}{t} & = & \lambda e^{(\lambda-1)\ln(t)} \\ y''(t) & = & \lambda e^{(\lambda-1)\ln(t)}(\lambda-1)\frac{1}{t} & = & \lambda(\lambda-1)e^{(\lambda-2)\ln(t)}. \end{array}$$

Plugging into the Euler ODE gives,

$$\lambda(\lambda-1)e^{\lambda\ln(t)} + p\lambda e^{\lambda\ln(t)} + qe^{\lambda\ln(t)} = (\lambda^2 + (p-1)\lambda + q)e^{\lambda\ln(t)}.$$

Therefore $y(t)$ is a solution of the ODE iff λ is a root of the quadratic polynomial,

$$Q(z) = z^2 + (p-1)z + q.$$

If $Q(z)$ has 2 distinct real roots r_1 and r_2, then 2 solutions of the ODE are $y_1(t) = t^{r_1}$ and $y_2(t) = t^{r_2}$. The Wronskian of this solution pair is,

$$W[y_1, y_2](t) = t^{r_1}(r_2t^{r_2-1}) - (r_1t^{r_1-1})t^{r_2} = (r_2 - r_1)t^{r_1+r_2-1} = (r_2 - r_1)t^{-p}$$

Because $r_2 - r_1 \neq 0$ and because $t^{-p} \neq 0$ for $t > 0$, the Wronskian is everywhere nonzero. Therefore $(y_1(t), y_2(t))$ is a basic solution set.

Next suppose that $Q(z)$ has two complex conjugate roots,

$$\lambda_\pm = \alpha \pm i\beta.$$

By Exercise 32, p. 185, the real and imaginary parts of the complex-valued solution $e^{\lambda_+\ln(t)}$ give real-valued solutions of the ODE. These are,

$$\begin{array}{rclcl} u(t) & = & e^{\alpha\ln(t)}\cos(\beta\ln(t)) & = & t^\alpha\cos(\beta\ln(t)) \\ v(t) & = & e^{\alpha\ln(t)}\sin(\beta\ln(t)) & = & t^\alpha\sin(\beta\ln(t)). \end{array}$$

The derivatives are,

$$\begin{array}{rclcl} u'(t) & = & \alpha t^{\alpha-1}\cos(\beta\ln(t)) + t^\alpha(-\sin(\beta\ln(t))\beta\frac{1}{t}) & = & t^{\alpha-1}(\alpha\cos(\beta\ln(t)) - \beta\sin(\beta\ln(t))) \\ v'(t) & = & \alpha t^{\alpha-1}\sin(\beta\ln(t)) + t^\alpha(+\cos(\beta\ln(t))\beta\frac{1}{t}) & = & t^{\alpha-1}(\alpha\sin(\beta\ln(t)) + \beta\cos(\beta\ln(t))) \end{array}$$

The Wronskian of this solution pair is,

$$t^\alpha\cos(\beta\ln(t))t^{\alpha-1}(\alpha\sin(\beta\ln(t)) + \beta\cos(\beta\ln(t))) - t^{\alpha-1}(\alpha\cos(\beta\ln(t)) - \beta\sin(\beta\ln(t)))t^\alpha\sin(\beta\ln(t)).$$

Simplifying,

$$W[u, v](t) = \beta t^{2\alpha-1}(\cos^2(\beta\ln(t)) + \sin^2(\beta\ln(t))) = \beta t^{2\alpha-1} = \beta t^{-p}$$

By hypothesis, $\beta \neq 0$. And for $t > 0$, $t^{-p} \neq 0$. So the Wronskian is everywhere nonzero. Therefore $(u(t), v(t))$ is a basic solution set.

(7)(10 points) p. 222, Problem 14

Solution, (a): Let L denote the 2^{nd} order linear differential operator,

$$L[y] = (D^2 + a(t)D + b(t))y.$$

Let $(y_1(t), y_2(t))$ be a solution pair of $L[y] = 0$. Differentiating,

$$W[y_1, y_2](t)' = (y_1(t)y_2'(t) - y_1'(t)y_2(t))' = (y_1'(t)y_2'(t) + y_1(t)y_2''(t) - y_1'(t)y_2'(t) - y_1''(t)y_2(t)).$$

Canceling like terms,

$$W' = y_2(t)(-y_1''(t)) - y_1(t)(-y_2''(t)).$$

For any solution of the homogeneous ODE,

$$-y''(t) = a(t)y'(t) + b(t)y(t).$$

Therefore,

$$W'(t) = y_2(t)(a(t)y_1'(t) + b(t)y_1(t)) - y_1(t)(a(t)y_2'(t) + b(t)y_2(t)).$$

Canceling and gathering terms,

$$W'(t) = -a(t)(y_1(t)y_2'(t) - y_1'(t)y_2(t)) = -a(t)W(t).$$

This is a first-order separable equation whose general solution is,

$$W(t) = W(t_0)\exp\left[-\int_{t_0}^{t} a(s)ds\right].$$

(b): Define the function $w(t)$ to be,

$$w(t) = \exp\left[-\int_{t_0}^{t} a(s)ds\right].$$

Observe that this is everywhere nonzero. Let $u(t)$ is a solution of $L[u] = 0$. Thus $u(t)$ and $w(t)$ are functions that *have already been specified* (the following equation is *not* an equation for $u(t)$ or $w(t)$). Let $v(t)$ be a solution of the first-order, linear inhomogeneous equation,

$$u(t)y' - u(t)y = w(t).$$

Assume that $v''(t)$ exists and is continuous.

By hypothesis, the Wronskian $W[u, v](t)$ equals $w(t)$. In particular, $W[u, v](t)$ satisfies the first-order ODE,

$$W[u, v]'(t) = w'(t) = -a(t)w(t) = -a(t)W[u, v](t).$$

Reversing the computations of (a), this leads to the equation,

$$u(t)v''(t) + (-u''(t))v(t) = W'(t) = -a(t)W(t) = -a(t)u(t)v'(t) + a(t)u'(t)v(t).$$

By hypothesis,

$$-u''(t) = a(t)u'(t) + b(t)u(t).$$

Substituting this into the equation above gives,

$$u(t)v''(t) + a(t)u'(t)v(t) + b(t)u(t)v(t) = -a(t)u(t)v'(t) + a(t)u'(t)v(t).$$

Canceling $a(t)u'(t)v(t)$ from each side of the equation and bringing all terms to the left side of the equation,

$$u(t)(v''(t) + a(t)v'(t) + b(t)v(t)) = 0.$$

Because $u(t)$ is not identically zero, on an open dense subset of the interval,

$$v''(t) + a(t)v'(t) + b(t)v(t) = 0.$$

A continuous function that is 0 on an open dense subset of an interval is 0 on the entire interval. Therefore $v(t)$ is a solution of $L[v] = 0$. Moreover the Wronskian $W[u, v](t) = w(t)$ is everywhere nonzero. Therefore $(u(t), v(t))$ is a basic solution pair of the 2^{nd} order linear ODE $L[y] = 0$.

(c): The normalized linear differential operator is,

$$L[y] = \left(D^2 - \frac{t+2}{t}D + \frac{2}{t}\right)y.$$

So $a(t) = -\frac{t+2}{t}$. Therefore,

$$w(t) = \exp\left[\int_{t_0}^{t}\left(1 + \frac{2}{s}\right)ds\right] = A\exp(t + 2\ln(t)),$$

for an appropriate choice of the constant A. Without loss of generality, set $A = -1$. Then $u(t)$ is a solution of the first order ODE,

$$e^t y' - e^t y = -t^2 e^t.$$

Canceling e^t from both sides of the equation,

$$y' - y = -t^2.$$

By the method of undetermined coefficients, the solution has the form $v(t) = a_2t^2 + a_1t + a_0$ for some choice of a_2, a_1, a_0. Plugging in gives the equation,

$$(-a_2)t^2 + (2a_2 - a_1)t + (a_1 - a_0) = -t^2.$$

The solution is $a_2 = 1$, $a_1 = 2$, $a_0 = 2$, i.e.,

$$v(t) = t^2 + 2t + 2.$$

Therefore a basic solution set is $(e^t, t^2 + 2t + 2)$ with Wronskian $W(t) = -t^2e^t$.

(8)(10 points) p. 223, Problem 17

Solution: By straightforward manipulation,

$$cy_d = y_1(t)\int_{t_0}^{t} \frac{-y_2(s)f(s)}{W[y_1,y_2](s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)f(s)}{W[y_1,y_2](s)}ds =$$
$$\int_{t_0}^{t} \frac{-y_1(t)y_2(s)f(s)}{W[y_1,y_2](s)}ds + \int_{t_0}^{t} \frac{y_2(t)y_1(s)f(s)}{W[y_1,y_2](s)}ds =$$
$$\int_{t_0}^{t} \frac{(y_1(s)y_2(t) - y_1(t)y_2(s))}{W[y_1,y_2](s)}f(s)ds.$$

Therefore,

$$y_d(t) = \int_{t_0}^{t} K(t,s)f(s)ds,$$

where,

$$K(t,s) = \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W[y_1,y_2](s)}.$$

It turns out that for any constant coefficient linear differential operator, the Green's kernel is,

$$K(t,s) = k(t-s)$$

for some continuously differentiable function $k(u)$. In particular, for $D^2 + 1$, a basic solution set is $(y_1, y_2) = (\cos(t), \sin(t))$. The Wronskian is $W[y_1,y_2](t) = 1$. So the Green's kernel is,

$$K(t,s) = \cos(s)\sin(t) - \cos(t)\sin(s) = \sin(t-s).$$

So a particular solution of $(D^2 + 1)y = f(t)$ is,

$$y_d(t) = \int_{t_0}^{t} \sin(t-s)f(s)ds.$$

This formula will reappear in the unit on the Laplace transform.

The equation $(D^2 - \frac{2}{t^2})y = 0$ is an Euler ODE for $p = 0$, $q = -2$. The associated quadratic equation is,

$$Q(z) = z^2 - z - 2 = (z-2)(z+1).$$

By Exercise 6, p. 221, a basic solution set is (t^{-1}, t^2). The Wronskian is 3. So the Green's kernel is,

$$K(t,s) = \frac{1}{3}(s^{-1}t^2 - t^{-1}s^2) = \frac{1}{3ts}(t^3 - s^3).$$

Observe that this is *not* of the form $k(t-s)$ for any function $k(u)$.