

## 18.034 SOLUTIONS TO PROBLEM SET 2

**Due date:** Friday, February 20 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

**Problem 1** (25 points) *The Implicit Function Theorem, Part 1.* In this problem you will apply the Contraction Mapping Fixed Point Theorem to prove the following theorem.

**Theorem 1.** Let  $(x_1, \dots, x_n, y)$  denote coordinates on  $\mathbb{R}^{n+1}$ . Let  $U$  be an open region in  $\mathbb{R}^{n+1}$ . Let  $f(x_1, \dots, x_n, y)$  be a continuous function on  $U$  such that the partial derivative  $\frac{\partial f}{\partial y}$  is defined and continuous on  $U$ . Let  $p = (\bar{x}_1, \dots, \bar{x}_n, \bar{y})$  be a point in  $U$  such that  $\frac{\partial f}{\partial y}(p) \neq 0$ . **Correction:** Also assume that  $f(p) = 0$  (or else the theorem is wrong!).

There exist numbers  $a_1 > 0, \dots, a_n > 0, b > 0$  such that the multi-interval,

$$I = [\bar{x}_1 - a_1, \bar{x}_1 + a_1] \times \cdots \times [\bar{x}_n - a_n, \bar{x}_n + a_n] \times [\bar{y} - b, \bar{y} + b],$$

is contained in  $U$ , and there exists a continuous function  $y(x_1, \dots, x_n)$  defined on the closed multi-interval,

$$J = [\bar{x}_1 - a_1, \bar{x}_1 + a_1] \times \cdots \times [\bar{x}_n - a_n, \bar{x}_n + a_n],$$

whose graph lies in  $I$  such that  $f(x_1, \dots, x_n, y(x_1, \dots, x_n)) = 0$ , i.e., the function  $f$  is 0 on the graph of  $y(x_1, \dots, x_n)$ .

Denote  $m = \frac{\partial f}{\partial y}(p)$ . The trick is to consider the function,

$$g(x_1, \dots, x_n, y) = y - \frac{1}{m} f(x_1, \dots, x_n, y).$$

For a function  $y(x_1, \dots, x_n)$ ,  $f(x_1, \dots, x_n, y) = 0$  iff  $g(x_1, \dots, x_n, y) = y$ . This suggests trying to find  $y$  as a fixed point of the mapping,

$$T(y) = z, \quad z(x_1, \dots, x_n) = g(x_1, \dots, x_n, y(x_1, \dots, x_n)).$$

But first we need to know on what complete metric space this mapping is defined, and we have to guarantee that  $T$  is a  $u$ -contraction mapping for some  $0 < u < 1$ .

To ease notation, for each sequence of numbers  $a_1 > 0, \dots, a_n > 0$ , denote by  $J(a_1, \dots, a_n)$  the multi-interval,

$$J(a_1, \dots, a_n) = [\bar{x}_1 - a_1, \bar{x}_1 + a_1] \times \cdots \times [\bar{x}_n - a_n, \bar{x}_n + a_n].$$

Similarly, for each sequence of numbers  $a_1 > 0, \dots, a_n > 0, b > 0$ , denote by  $I(a_1, \dots, a_n, b)$  the multi-interval,

$$I(a_1, \dots, a_n, b) = [\bar{x}_1 - a_1, \bar{x}_1 + a_1] \times \cdots \times [\bar{x}_n - a_n, \bar{x}_n + a_n] \times [\bar{y} - b, \bar{y} + b].$$

**(a)** (5 points) Write down a careful argument that there exist numbers  $a_1^{\text{pre}} > 0, \dots, a_n^{\text{pre}} > 0, b > 0$  such that the multi-interval  $I(a_1^{\text{pre}}, \dots, a_n^{\text{pre}}, b)$  is contained in  $U$  and such that  $|\frac{\partial g}{\partial y}| \leq u$  everywhere on  $I(a_1^{\text{pre}}, \dots, a_n^{\text{pre}}, b)$ .

**Solution:** The region  $U$  is open, hence there exists a radius  $r > 0$  such that the open ball of radius  $r$  centered on  $p$  is contained in  $U$ . By definition of  $g$ ,

$$\frac{\partial g}{\partial y}(p) = 1 - \frac{1}{m} \frac{\partial f}{\partial y}(p) = 1 - 1 = 0.$$

So for any  $q$  in  $U$ ,

$$\left| \frac{\partial g}{\partial y}(q) \right| = \left| \frac{\partial g}{\partial y}(q) - \frac{\partial g}{\partial y}(p) \right|.$$

Because  $\frac{\partial g}{\partial y}$  is continuous, for  $\epsilon = u$ , there exists a radius  $0 < \delta \leq r$  such that if  $d(p, q) < \delta$ , then  $|\frac{\partial g}{\partial y}(q) - \frac{\partial g}{\partial y}(p)| < \epsilon$ . Let  $a$  be any positive number with  $0 < a < \frac{1}{\sqrt{n+1}}\delta$  and define,

$$a_1^{\text{pre}} := a, \dots, a_n^{\text{pre}} := a, b := a.$$

Then for every point  $q = (x_1, \dots, x_n, y)$  in  $I(a_1^{\text{pre}}, \dots, a_n^{\text{pre}}, b)$ , each  $|x_i - \bar{x}_i| \leq a < \frac{1}{\sqrt{n+1}}\delta$ , and  $|y - \bar{y}| \leq a < \frac{1}{\sqrt{n+1}}\delta$ . Therefore,

$$d(p, q)^2 = \sum_{i=1}^n (x_i - \bar{x}_i)^2 + (y - \bar{y})^2 < (n+1) \left( \frac{1}{\sqrt{n+1}}\delta \right)^2 = \delta^2.$$

Since  $d(p, q) < \delta$ ,  $|\frac{\partial g}{\partial y}(q) - \frac{\partial g}{\partial y}(p)| < u$ .

**(b)**(5 points) Write down a careful argument that there exist numbers  $0 < a_1 \leq a_1^{\text{pre}}, \dots, 0 < a_n \leq a_n^{\text{pre}}$  such that  $|g(x_1, \dots, x_n, \bar{y}) - \bar{y}| \leq (1-u)b$  on  $J(a_1, \dots, a_n)$ . Conclude that the mapping on  $I(a_1, \dots, a_n, b)$  given by,

$$(x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n, g(x_1, \dots, x_n, y)),$$

maps  $I(a_1, \dots, a_n, b)$  back into itself. Therefore, if  $y(x_1, \dots, x_n)$  is a continuous function on  $J(a_1, \dots, a_n)$  whose graph lies in  $I(a_1, \dots, a_n, b)$ , then also the graph of

$$z(x_1, \dots, x_n) = g(x_1, \dots, x_n, y(x_1, \dots, x_n)),$$

lies in  $I(a_1, \dots, a_n, b)$ . (**Hint:** Use the mean value theorem to prove that  $|g(x_1, \dots, x_n, y) - g(x_1, \dots, x_n, \bar{y})| \leq ub$ .)

**Solution:** Denote  $p' = (\bar{x}_1, \dots, \bar{x}_n)$ . Define the function  $h$  on  $J(a_1^{\text{pre}}, \dots, a_n^{\text{pre}})$  by,

$$h(x_1, \dots, x_n) = g(x_1, \dots, x_n, \bar{y}) = \bar{y} - \frac{1}{m}f(x_1, \dots, x_n, \bar{y}).$$

Because  $g$  is continuous,  $h$  is continuous. Because  $f(p) = 0$ ,

$$h(p') = g(p) = \bar{y}.$$

Because  $h$  is continuous, for  $\epsilon' = (1-u)b$ , there exists a radius  $0 < \delta' < a$  such that for all  $q' \in J(a_1^{\text{pre}}, \dots, a_n^{\text{pre}})$ , if  $d(p', q') < \delta'$ , then  $|h(q') - h(p')| < \epsilon'$ . Let  $a'$  be any number with  $0 < a' < \frac{1}{\sqrt{n+1}}\delta'$ , and define,

$$a_1 := a', \dots, a_n := a'.$$

As in part (a), for all  $q' \in J(a_1, \dots, a_n)$ ,  $d(q', p') < \delta'$  holds. Hence,

$$|g(x_1, \dots, x_n, \bar{y}) - \bar{y}| < (1-u)b.$$

Let  $q = (x_1, \dots, x_n, y)$  be a point in  $I(a_1, \dots, a_n, b)$ . Then,

$$|g(x_1, \dots, x_n, y) - y| \leq |g(x_1, \dots, x_n, y) - g(x_1, \dots, x_n, \bar{y})| + |g(x_1, \dots, x_n, \bar{y}) - \bar{y}|.$$

By the last paragraph, the second term is less than  $(1-u)b$ . By the mean value theorem, there exists  $c$  between  $y$  and  $\bar{y}$  such that,

$$g(x_1, \dots, x_n, y) - g(x_1, \dots, x_n, \bar{y}) = \frac{\partial g}{\partial y}(x_1, \dots, x_n, c)(y - \bar{y}).$$

Because  $|\frac{\partial g}{\partial y}| < u$  on  $I(a_1, \dots, a_n, b)$ ,

$$|g(x_1, \dots, x_n, y) - g(x_1, \dots, x_n, \bar{y})| \leq u|y - \bar{y}| \leq ub.$$

Putting both inequalities together,  $|g(x_1, \dots, x_n, y) - y| < (1 - u)b + ub = b$ . So the mapping,

$$(x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n, g(x_1, \dots, x_n)),$$

sends  $I(a_1, \dots, a_n, b)$  into itself. In particular, if  $y(x_1, \dots, x_n)$  is a continuous function on  $J(a_1, \dots, a_n)$  whose graph lies in  $I(a_1, \dots, a_n, b)$ , then also,

$$z(x_1, \dots, x_n) = g(x_1, \dots, x_n, y(x_1, \dots, x_n)),$$

is a continuous function on  $J(a_1, \dots, a_n)$  whose graph lies in  $I(a_1, \dots, a_n, b)$ .

(c)(5 points) Define  $B$  to be the metric space of continuous functions  $y$  on  $J(a_1, \dots, a_n)$  whose graph lies in  $I(a_1, \dots, a_n, b)$  that satisfy  $y(\bar{x}_1, \dots, \bar{x}_n) = \bar{y}$ . The *uniform metric* (often also called the  $L^\infty$  metric, or the *sup metric*) is defined by,

$$d(y_1, y_2) = \max_{t \in J} |y_1(t) - y_2(t)|.$$

Use the natural multi-dimensional generalization of the Cauchy Test to prove that this is a complete metric space. You need not *prove* the multi-dimensional generalization! Simply write down a careful statement of what you believe the generalization says, and apply this appropriately to deduce that  $B$  is a complete metric space.

**Solution:** The multi-variable version of the Cauchy Test says the same thing as the one-variable version.

**Theorem 2** (Cauchy Test). *Let  $R \subset \mathbb{R}^n$  be a closed, bounded region, and let  $(y_i)$  be a sequence of continuous functions on  $R$  that is Cauchy with respect to the uniform metric. Then there exists a continuous function  $y$  on  $R$  such that  $(y_i)$  converges to  $y$  with respect to this metric.*

The theorem also holds if  $B$  is not a closed, bounded region, but one has to change the definition of  $d(y_1, y_2)$  because continuous functions do not necessarily attain their maximum. The proof is identical to the proof of the one-variable version. Because  $[\bar{y} - b, \bar{y} + b]$  is a closed interval, it contains all of its limits. Thus if the graph of each function  $y_i$  is in  $I(a_1, \dots, a_n, b)$ , then for every  $q' = (x_1, \dots, x_n)$  in  $J(a_1, \dots, a_n)$ , the limit  $y(q')$  of the sequence  $(y_i(q'))$  is in  $[\bar{y} - b, \bar{y} + b]$ . Therefore the graph of  $y$  is contained in  $I(a_1, \dots, a_n, b)$ . So a Cauchy sequence in  $B$  converges to a limit in  $B$ , i.e.  $B$  is a complete metric space.

(d)(5 points) Prove that for each continuous function  $y$  in  $B$ , the following function  $z$  is also in  $B$ ,

$$z(x_1, \dots, x_n) = g(x_1, \dots, x_n, y(x_1, \dots, x_n)).$$

Therefore the mapping  $T(y) = z$  is a mapping from  $B$  into itself.

**Solution:** This is really just the last part of (b).

(e)(5 points) Prove that  $T$  is a  $u$ -contraction mapping. Use the Contraction Mapping Fixed Point Theorem to deduce that there exists a continuous function  $y$  in  $B$  such that  $T(y) = y$ . Deduce that  $f$  is 0 on the graph of  $y$ . (**Hint:** Use the mean value theorem.)

**Solution:** Let  $y_1$  and  $y_2$  be functions in  $B$ , and denote  $D := d(y_1, y_2)$ . Denote  $z_i(x_1, \dots, x_n, y_i(x_1, \dots, x_n))$  for  $i = 1, 2$ . Let  $q' = (x_1, \dots, x_n)$  be an element of  $J(a_1, \dots, a_n)$ . By the mean value theorem, there exists a number  $c$  between  $y_1(q')$  and  $y_2(q')$  such that,

$$z_1(q') - z_2(q') = g(x_1, \dots, x_n, y_1(q')) - g(x_1, \dots, x_n, y_2(q')) = \frac{\partial g}{\partial y}(x_1, \dots, x_n, c)(y_1(q') - y_2(q')).$$

Because  $|\frac{\partial g}{\partial y}| < u$  on  $I(a_1, \dots, a_n, b)$ ,

$$|z_1(q') - z_2(q')| \leq u|y_1(q') - y_2(q')| \leq uD.$$

Therefore,

$$d(z_1, z_2) = \max_{q' \in J} |z_1(q') - z_2(q')| \leq ud(y_1, y_2).$$

Therefore  $T(y) = z$  is a  $u$ -contraction mapping theorem.

Because  $B$  is a complete metric space and  $T$  is a  $u$ -contraction mapping, by the Contraction Mapping Fixed Point Theorem there exists a unique fixed point  $y$  of  $T$ . Because  $T(y) = y$ , for every  $q' \in J(a_1, \dots, a_n)$ ,

$$y(q') = g(q', y(q')) = y(q') - \frac{1}{m}f(q', y(q')).$$

Therefore  $f(q', y(q')) = 0$  for every  $q' \in J(a_1, \dots, a_n)$ , i.e.,  $f$  is zero on the graph of  $y$ .

**Problem 2**(5 points) *The Implicit Function Theorem, Part II.* The notation is from Problem 1. Let  $(x_1, \dots, x_n, y)$  be a point in  $I$  such that  $f(x_1, \dots, x_n, y) = 0$ . Prove that  $y = y(x_1, \dots, x_n)$ . Therefore the points in  $I$  where  $f$  is 0 are exactly the points on the graph of  $y(x_1, \dots, x_n)$ . (**Hint:** If  $y \neq y(x_1, \dots, x_n)$ , use the mean value theorem to find a number  $y_1$  between  $y$  and  $y(x_1, \dots, x_n)$  where the derivative  $\frac{\partial f}{\partial y}(x_1, \dots, x_n, y_1)$  gives a contradiction.)

**Solution:** I apologize for the poor notation of this problem: using  $y$  both to denote a function, and the coordinate of a supposedly different point. Denote the point by  $(x_1, \dots, x_n, w)$  instead. Denote  $q' = (x_1, \dots, x_n)$ . By the mean value theorem, there is a number  $y_1$  between  $w$  and  $y(q')$  such that,

$$f(q', w) - f(q', y(q')) = \frac{\partial f}{\partial y}(q', y_1)(w - y(q')).$$

Since  $f(q', w) = f(q', y(q')) = 0$ , either  $w = y(q')$  or  $\frac{\partial f}{\partial y}(q', y_1) = 0$ . By definition of  $g$ ,

$$\frac{\partial f}{\partial y} = m - m \frac{\partial g}{\partial y}.$$

Since  $|\frac{\partial g}{\partial y}| < u$  on  $I(a_1, \dots, a_n, b)$ ,

$$\left| \frac{\partial f}{\partial y} - m \right| \leq |m|u < |m|.$$

So  $\frac{\partial f}{\partial y}$  is nonzero at every point of  $I(a_1, \dots, a_n, b)$ . Therefore  $w = y(q')$ .

**Problem 3**(10 points) Appendix A.1, Problem 1, p. 677.

**Solution, (a):** The function  $|y|$  is not differentiable on any interval containing 0, because the slope of the graph as  $y \rightarrow 0^-$  is  $-1$ , and the slope of the graph as  $y \rightarrow 0^+$  is  $+1$ . So, on any region that intersects the  $t$ -axis, it is not true that  $\frac{\partial f}{\partial y}$  is everywhere defined.

**(b):** One solution of the IVP is  $y(t) = 0$  for all  $t$ . Let  $z(t)$  be any solution of the IVP. Suppose that there exists  $t > t_0$  such that  $z(t) \neq 0$ . There exists a largest  $t_1$  with  $t_0 \leq t_1 < t$  such that  $z(t) = 0$  for all  $t \in [t_0, t_1]$ . Let  $t_2$  be a number with  $t_1 < t_2 < t$  such that  $t_2 - t_1 < 1$ . Denote by  $M$ ,

$$M = \max_{t \in [t_1, t_2]} |z(t)|.$$

Because there exists  $t \in [t_1, t_2]$  with  $z(t) \neq 0$ ,  $M > 0$ . On the other hand, for all  $t \in [t_1, t_2]$ ,

$$z(t) = z(t_1) + \int_{t_1}^t z'(s)ds = 0 + \int_{t_1}^t |z(s)|ds.$$

Therefore,

$$|z(t)| = z(t) = \int_{t_1}^t |z(s)| ds \leq \int_{t_1}^t M ds = M(t - t_1) \leq M(t_2 - t_1) < M.$$

Therefore  $\max_t |z(t)| < M$ , i.e.  $M < M$ . This is absurd. Therefore the original hypothesis is false, i.e.  $z(t) = 0$  for all  $t > t_0$ . A very similar argument proves that  $z(t) = 0$  for all  $t < t_0$ .

No contradiction exists. The uniqueness theorem is not an “if and only if” theorem. If the hypotheses are satisfied, the conclusion of the theorem holds. But, as this example illustrates, the conclusion of the theorem may hold even if the hypotheses are not satisfied.

(c): In fact the proof on p. 676 exactly proves this, so there is no need to repeat the proof. The function  $|y|$  satisfies a Lipschitz condition with constant  $L = 1$ :  $||a| - |b|| \leq |a - b|$ . Therefore, if a solution to the IVP exists (which it clearly does because  $y(t) = 0$  is a solution), the solution is unique.

**Problem 4**(5 points) Section 2.4, Problem 16, p. 67 (just draw a rough sketch; the definition of “Step” is on the inside front cover of the text).

**Solution:** On the interval  $(-\infty, 0)$ , the differential equation is  $y' = 0$ . The general solution of this ODE is,

$$y_{<0}(t) = C.$$

On the interval  $(0, \infty)$ , the differential equation is  $y' = y$ . The general solutions of this ODE is

$$y_{>0}(t) = Ae^t.$$

The matching condition at  $t = 0$  for  $y_{<0}$  and  $y_{>0}$  to extend to a continuous function on all of  $(-\infty, \infty)$  is,

$$\lim_{t \rightarrow 0^-} y_{<0}(t) = \lim_{t \rightarrow 0^+} y_{>0}(t), \text{ i.e., } C = A.$$

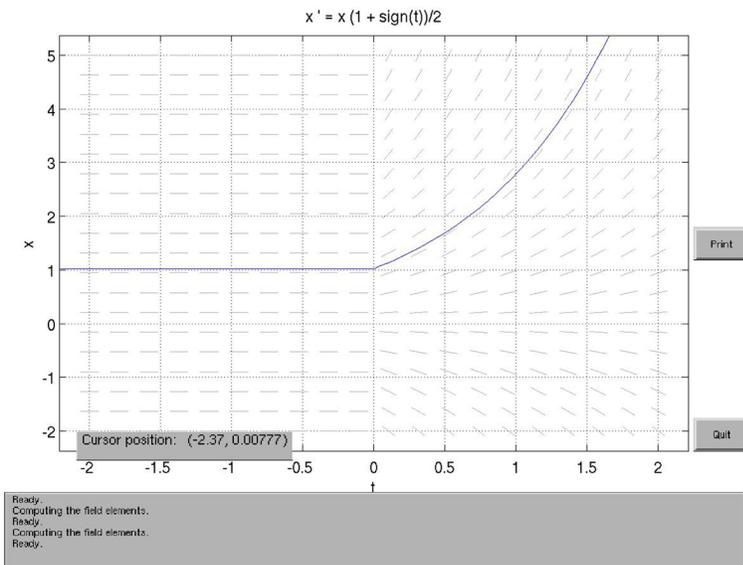
Therefore the generalized solutions of the ODE are,

$$y(t) = \begin{cases} C, & t \leq 0 \\ Ce^t, & t > 0 \end{cases}$$

In particular, the generalized solution of the IVP is,

$$y(t) = \begin{cases} 1, & t \leq 0 \\ e^t, & t > 0 \end{cases}$$

The graph of this function can be found [here](#).



**Problem 5** (5 points) Section 2.4, Problem 17, p. 67 (as above, just draw a rough sketch).

**Solution:** There is more than one way to solve this problem. One method is to apply Theorem 2.2.1, p. 50 with the given piecewise continuous input. This does give a generalized solution to the IVP, despite the fact that  $q(t)$  does not satisfy the hypotheses of the theorem (yet another illustration that many of our theorems/algorithms are more robust than the “clean formulation” implies). Another solution, the one I will follow, is to apply the same reasoning as in the last problem. I personally prefer this approach whenever it is reasonable, because it applies to a broader class of ODEs than first-order linear ODEs.

On the interval  $(-\infty, 1)$ , the ODE is  $y' + 2y = 1$ . An integrating factor is  $e^{2t}$ , leading to the ODE

$$(e^{2t}y)' = e^{2t}.$$

The general solution is,

$$e^{2t}y = \frac{1}{2}e^{2t} + C, \quad y = \frac{1}{2} + Ce^{-2t}.$$

On the interval  $(1, \infty)$ , the ODE is  $y' + 2y = 0$ . Using the same integrating factor, the general solution is,

$$e^{2t}y = A, \quad y = Ae^{-2t}.$$

The matching condition at  $t = 1$  is  $\frac{1}{2} + Ce^{-2} = Ae^{-2}$ . Thus the generalized solutions are,

$$y(t) = \begin{cases} \frac{1}{2} + Ce^{-2t}, & t \leq 1 \\ (\frac{e^2}{2} + C)e^{-2t}, & t > 1 \end{cases}$$

In particular, the generalized solution of the IVP is,

$$y(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2t}, & t \leq 1 \\ \frac{1}{2}(e^2 - 1)e^{-2t}, & t > 1 \end{cases}$$

The graph of this function can be found [here](#).

