18.034 SOLUTIONS TO PROBLEM SET 10

Due date: Friday, May 7 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Each of the following problems is from the textbook. The point value of the problem is next to the problem.

(1)(10 points) p. 463, Problem 21

Solution: The solution of the IVP,

$$\begin{cases} x' = -x^3, \\ y' = 1, \\ x(0) = x_0, \\ y(0) = y_0 \end{cases}$$

is given by,

$$\begin{cases} x(t) &= x_0/\sqrt{1+2x_0^2 t}, \\ y(t) &= y_0 + t \end{cases}$$

and the maximally extended solution is defined on the interval

$$I = (-1/2x_0^2, \infty).$$

If $-T \in I$, then the pair $(x(t-T), y(t-T))$ is again a solution corresponding to the initial point,

$$\begin{cases} x(0) &= x_0/\sqrt{1-2x_0^2 T}, \\ y(0) &= y_0 - T \end{cases}$$

(2)(10 points) p. 463, Problem 22

Solution: The solution of the IVP,

$$\begin{cases} x' = x^2(1+y), \\ y' = -y, \\ x(0) = x_0, \\ y(0) = y_0 \end{cases}$$

is given by,

$$\begin{cases} x(t) &= x_0/(1 + x_0 t + x_0 y_0(1-e^{-t})), \\ y(t) &= y_0 e^{-t} \end{cases}$$

The maximally extended solution is defined on an interval $I = (a, b)$ where a and b are solutions of the equation,

$$x_0 y_0 (1 - e^{-t}) + x_0 t + 1 = 0,$$

and a is the largest negative solution (or $-\infty$) and b is the smallest positive solution (or ∞).

If $-T \in I$, then the pair $(x(t-T), y(t-T))$ is again a solution corresponding to the initial point,

$$\begin{cases} x(0) &= x_0/(1 - x_0 T + x_0 y_0(1-e^{T})), \\ y(0) &= y_0 e^{T} \end{cases}$$

(3)(10 points) p. 463, Problem 33

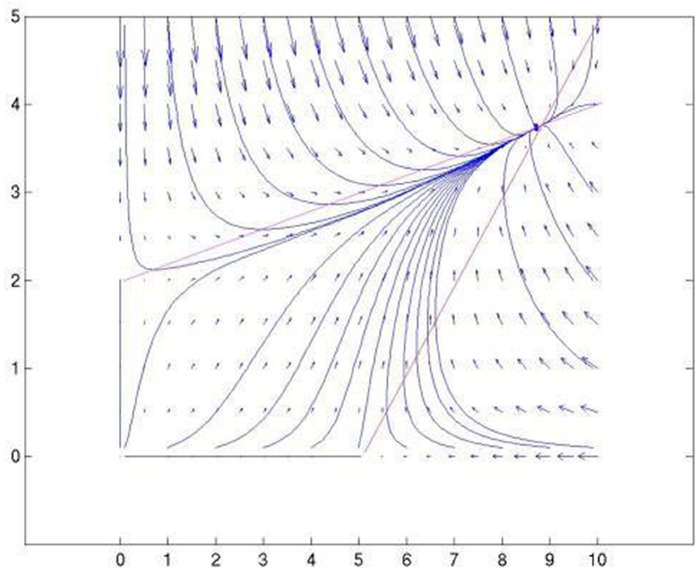
Solution: In fact, the hint *is* the solution of the problem. It should be mentioned, due to the theorem about maximally extended solutions, the maximally extended solution is defined on an interval $[0, T')$ where $T' > T$: in fact if T' is finite, then

$$\lim_{t \to T'} x(t) \in \partial S,$$

in the sense that the limit exists and is a boundary point of S. Because p is not a boundary point of S, it follows that $T' > T$.

(4)(10 points) p. 472, Problem 15

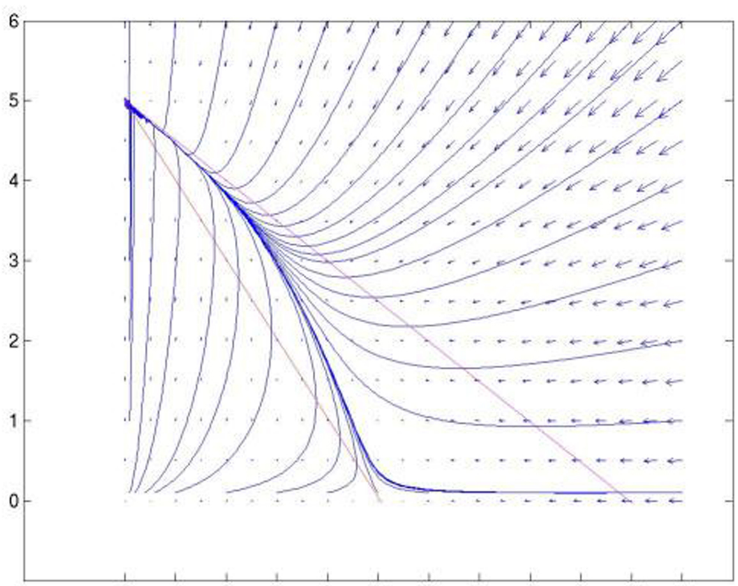
Solution: This is a cooperation model with overcrowding. A JPEG of the direction field is given below.



Exercise 15, p. 472

(5)(10 points) p. 472, Problem 17

Solution: This is a competition model with overcrowding. For almost all initial points, Species x becomes extinct and Species y approaches the equilibrium value $y = 5$. A JPEG of the direction field is given below.



Exercise 17, p. 472