18.034 SOLUTIONS TO PROBLEM SET 1

Due date: Friday, February 13 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Problem 1(20 points) The logistic model for a fish population with harvesting (p. 17) leads to the following IVP:

$$\begin{cases} y' = ay - cy^2 - H, \\ y(0) = y_0 \end{cases}$$

Here a and y_0 are positive and c and H are nonnegative. The IVP is defined on the interval $(0, \infty)$. Also, the model is only valid as long as $y(t) \geq 0$: If at any instant t_1 (greater than 0) $y(t_1)$ equals 0, then the population is extinct, and the population will remain extinct for all $t \geq t_1$.

(a)(10 points) The *equilibrium solutions* are the solutions of the ODE (without the initial condition) for which $y'(t) = 0$ for all t. Find inequalities among a, c, and H that determine when there will be 2 equilibrium solutions, 1 equilibrium solution, or no equilibrium solutions.

Solution: The equilibrium solutions are the constants y such that $ay - cy^2 - H = 0$; the normal form is $-cy^2 + ay - H = 0$. The discriminant of this quadratic equation is $(a)^2 - 4(-c)(-H) = a^2 - 4cH$. By the quadratic formula, the number of solutions is,

$$\begin{cases} 2, & \text{both } c \neq 0 \text{ and } a^2 - 4cH > 0 \\ 1, & \text{either } \left(c \neq 0 \text{ and } a^2 - 4cH = 0\right), \text{ or } c = 0 \\ 0, & a^2 - 4cH < 0. \end{cases}$$

(b)(10 points) Suppose that both a and c are positive. What is the maximum value of H for which there is an equilibrium solution? If H is larger than this value, what is the long-term behavior of any solution of the ODE?

Solution: By part (a), the maximum value of H is $H_0 = \frac{a^2}{4c}$. If $H > H_0$, then $y' = ay - cy^2 - H$ is negative for all values of y. Therefore the solution is everywhere decreasing.

Let's be more precise. Completing the square gives,

$$ay - cy^2 - H = -c\left(y - \frac{a}{2}\right)^2 - \left(H - \frac{a^2}{4c}\right).$$

Therefore, y' is at most $-(H - \frac{a^2}{4c})$. Denote $z(t) = -(H - \frac{a^2}{4c})t + y_0$. Then $y' - z'$ is at most 0, i.e., $y - z$ is nonincreasing. Also $y(0) - z(0) = 0$. Therefore $y - z$ is nonpositive. So $y(t) \leq z(t)$. Therefore, the population becomes extinct at a time,

$$t \leq \frac{4cy_0}{4cH - a^2}.$$

In fact this understates the truth – if you solve the separable differential equation exactly you will find there is a time $\tau(a, c, H) > 0$ so that, independent of the initial value y_0, the population becomes extinct at a time $t \leq \tau$.

Problem 2(20 points) After a change of variables, the logistic equation with harvesting reduces to the following IVP (neglecting the extinction issue),

$$\begin{cases} x' = -x^2 + K, \\ x(0) = x_0 > 0 \end{cases}$$

where $x = x(t)$ and where K is a constant. Suppose that $K = b^2$ for some $b > 0$.

(a)(10 points) Formally rewrite the ODE as $f(x)dx = g(t)dt$ and integrate to find an exact solution. Express your answer in the form $b - x = h(t)$ for some expresion $h(t)$. Don't forget the special case $x_0 = b$.

The ODE separates as,

$$\int \frac{1}{b^2 - x^2} dx = \int dt.$$

By partial fractions, this is the same as,

$$\int \left(\frac{1}{b + x} + \frac{1}{b - x} \right) dx = \int 2b dt.$$

Antidifferentiating,

$$\ln \left(\frac{b + x}{b - x} \right) = 2bt + C.$$

Exponentiating,

$$\frac{b + x}{b - x} = A' e^{2bt},$$

or equivalently,

$$\frac{b - x}{b + x} = A e^{-2bt}.$$

Rewriting $b + x = 2b - (b - x)$, and solving for $b - x$ gives,

$$b - x(t) = \frac{2bAe^{-2bt}}{1 + Ae^{-2bt}}.$$

If $x_0 \neq b$, define a new parameter $\alpha = \frac{b - x_0}{2b}$. Then, solving in terms of α,

$$(b - x(t)) = \begin{cases} (b - x_0)e^{-2bt} \left(\frac{1}{(1-\alpha)+\alpha e^{-2bt}} \right), & x_0 \neq b, \\ 0, & x_0 = b. \end{cases}$$

(b)(10 points) At some instant t_1, the value of $x(t_1)$ is very close to b. At that instant, the value of b in the differential equation is abruptly increased to a larger value b_1, and $x(t)$ gradually moves from the value b to the value b_1. Assuming $b_1 - b$ is small compared to b, approximately how much time τ elapses before the difference $b_1 - x(t_1 + \tau)$ is one half of the initial difference $b_1 - b$?

Solution: To simplify the problem, change coordinates in t so that $t_1 = 0$. Because the ODE is autonomous, this doesn't change the ODE (this will be the key to analyzing solutions of autonomous ODEs later on). Let $x_0 = x(t_1)$. Then the solution of the IVP with b_1 has the form,

$$(b_1 - x(t)) = (b_1 - x_0)e^{-2b_1 t} \left(\frac{1}{(1 - \alpha_1) + \alpha_1 e^{-2b_1 t}} \right),$$

where $\alpha_1 = \frac{b - x_0}{2b}$.

By hypothesis, $\alpha \approx 0$. Therefore the third factor in the solution is approximately 1, and the solution of the IVP is approximately a decreasing exponential,

$$(b_1 - x(t)) \approx (b_1 - x_0)e^{-2b_1 t}.$$

So the half-life is

$$\tau \approx \frac{\ln(2)}{2b_1}.$$

(c)(0 points – not to be written up/handed in). Critical ecosystem double whammy. Interpret your answer from (b). In particular, if the parameters a, c and H are near the critical value for extinction, does the system respond more quickly or less quickly to a decrease in H than if the parameters are far from the critical value?

Solution: This part was not to be handed in. The "solution" is only given for fun. The change of variables necessary to put the ODE in standard form is,

$$\begin{cases} x = c\left(y - \frac{a}{2}\right), \\ b = \sqrt{\frac{a^2}{4} - cH}. \end{cases}$$

So if a, c and H are near the critical value, then b is near 0. Decreasing H while holding a and c fixed increases b to a new value b_1. By (b), the half-life, or "reaction time", of the system to this change is proportional to

$$\frac{1}{b_1} \approx \frac{1}{b}.$$

So when b is small, the reaction time is large. This is the "double whammy": not only is the population close to the critical value of extinction (so a natural disaster, etc. could easily drive the population to extinction), but also a positive change in the environment (for instance, a government ban on fishing in a certain area) takes a long time to have a positive impact on the population.

Problem 3(5 points) Exercise 14, p. 49.

Solution: It is easier to spot the integrating factor without putting the ODE in normal form. For any ODE of the form,

$$ty' + ay = q(t), \ t \geq 0,$$

an integrating factor is clearly $u(t) = t^{a-1}$,

$$(t^a y)' = t^{a-1}q(t).$$

In this case, antidifferentiating both sides,

$$t^2 y(t) = \int t^3 dt = \frac{1}{4}t^4 + C.$$

So the general solution is,

$$y(t) = \frac{1}{4}t^2 + \frac{C}{t^2}, \quad t \geq 0.$$

The qualitative behavior as $t \to 0^+$ depends on the constant C. If $C > 0$, then $y(t)$ diverges to $+\infty$ as $\frac{1}{t^2}$. If $C = 0$, then $y(t)$ converges to 0 as t^2. If $C < 0$, then $y(t)$ diverges to $-\infty$ as $\frac{-1}{t^2}$.

The qualitative behavior as $t \to \infty$ is the same for all solutions: the graph of $y(t)$ converges to the graph of the *steady-state solution*, $\frac{1}{4}t^2$. In particular it diverges to ∞ as t^2.

Problem 4(5 points) Exercise 20, p. 49.

As above, the integrating factor is easier to “eyeball” than to deduce formally. Multiplying both sides of the equation by $\sin t$ gives,

$$(\sin t)y' + (\cos t)y = 2(\sin t)(\cos t), \quad y(3\pi/4) = 2.$$

This is the same as,

$$(\sin(t)y)' = (\sin(t)^2)', \quad y(3\pi/4) = 2.$$

Antidifferentiating, the general solution is,

$$\sin(t)y = \sin(t)^2 + C.$$

Solving the initial condition, $C = -3$. So the solution of the IVP is,

$$y(t) = \sin(t) - 3\csc(t).$$

Because $\sin(t) \to 0$ like t as $t \to 0^+$, $y(t)$ diverges to $-\infty$ like $\frac{-1}{t}$ as $t \to 0^+$.