18.034 PROBLEM SET 2

Due date: Friday, February 20 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Problem 1(25 points) *The Implicit Function Theorem, Part 1.* In this problem you will apply the Contraction Mapping Fixed Point Theorem to prove the following theorem.

Theorem 1. *Let $(x_1, ..., x_n, y)$ denote coordinates on $\mathbb{R}^{n+1}$. Let U be an open region in $\mathbb{R}^{n+1}$. Let $f(x_1, \dots, x_n, y)$ be a continuous function on U such that the partial derivative $\frac{\partial f}{\partial y}$ is defined and continuous on U. Let $p = (\overline{x}_1, \dots, \overline{x}_n, \overline{y})$ be a point in U such that $\frac{\partial f}{\partial y}(p) \neq 0$.*

There exist numbers $a_1 > 0, \dots, a_n > 0$, $b > 0$ such that the multi-interval,

$$I = [\overline{x}_1 - a_1, \overline{x}_1 + a_1] \times \cdots \times [\overline{x}_n - a_n, \overline{x}_n + a_n] \times [\overline{y} - b, \overline{y} + b],$$

is contained in U, and there exists a continuous function $y(x_1, \dots, x_n)$ defined on the closed multi-interval,

$$J = [\overline{x}_1 - a_1, \overline{x}_1 + a_1] \times \cdots \times [\overline{x}_n - a_n, \overline{x}_n + a_n],$$

whose graph lies in I such that $f(x_1, \dots, x_n, y(x_1, \dots, x_n)) = 0$, i.e., the function f is 0 on the graph of $y(x_1, \dots, x_n)$.

Denote $m = \frac{\partial f}{\partial y}(p)$. The trick is to consider the function,

$$g(x_1, \dots, x_n, y) = y - \frac{1}{m} f(x_1, \dots, x_n, y).$$

For a continuous function $y(x_1, \dots, x_n)$, $f(x_1, \dots, x_n, y) = 0$ iff $g(x_1, \dots, x_n, y) = y$. This suggest trying to find y as a fixed point of the mapping,

$$T(y) = z, \quad z(x_1, \dots, x_n) = g(x_1, \dots, x_n, y(x_1, \dots, x_n)).$$

But first we need to know on what complete metric space this mapping is defined, and we have to guarantee that T is a u-contraction mapping for some $0 < u < 1$.

To ease notation, for each sequence of numbers $a_1 > 0, \dots, a_n > 0$, denote by $J(a_1, \dots, a_n)$ the multi-interval,

$$J(a_1, \dots, a_n) = [\overline{x}_1 - a_1, \overline{x}_1 + a_1] \times \cdots \times [\overline{x}_n - a_n, \overline{x}_n + a_n].$$

Similarly, for each sequence of numbers $a_1 > 0, \dots, a_n > 0$, $b > 0$, denote by $I(a_1, \dots, a_n, b)$ the multi-interval,

$$I(a_1, \dots, a_n, b) = [\overline{x}_1 - a_1, \overline{x}_1 + a_1] \times \cdots \times [\overline{x}_n - a_n, \overline{x}_n + a_n] \times [\overline{y} - b, \overline{y} + b].$$

(a)(5 points) Write down a careful argument that there exist numbers $a_1^{\text{pre}} > 0, \dots, a_n^{\text{pre}} > 0, b > 0$ such that the multi-interval $I(a_1^{\text{pre}}, \dots, a_n^{\text{pre}}, b)$ is contained in U and such that $|\frac{\partial g}{\partial y}| \leq u$ everywhere on $I(a_1^{\text{pre}}, \dots, a_n^{\text{pre}}, b)$.

(b)(5 points) Write down a careful argument that there exist numbers $0 < a_1 \leq a_1^{\text{pre}}, \dots, 0 < a_n \leq a_n^{\text{pre}}$ such that $|g(x_1, \dots, x_n, \overline{y}) - \overline{y}| \leq (1 - u)b$ on $J(a_1, \dots, a_n)$. Conclude that the mapping on $I(a_1, \dots, a_n, b)$ given by,

$$(x_1, \dots, x_n, y) \mapsto (x_1, \dots, x_n, g(x_1, \dots, x_n, y)),$$

maps $I(a_1, \ldots, a_n, b)$ back into itself. Therefore, if $y(x_1, \ldots, x_n)$ is a continuous function on $J(a_1, \ldots, a_n)$ whose graph lies in $I(a_1, \ldots, a_n, b)$, then also the graph of

$$z(x_1, \ldots, x_n) = g(x_1, \ldots, x_n, y(x_1, \ldots, x_n)),$$

lies in $I(a_1, \ldots, a_n, b)$. (**Hint**: Use the mean value theorem to prove that $|g(x_1, \ldots, x_n, y) - g(x_1, \ldots, x_n, \overline{y})| \leq ub$.)

(c)(5 points) Define B to be the metric space of continuous functions y on $J(a_1, \ldots, a_n)$ whose graph lies in $I(a_1, \ldots, a_n, b)$ that satisfy $y(\overline{x}_1, \ldots, \overline{x}_n) = \overline{y}$. The metric is defined by,

$$d(y_1, y_2) = \max_{t \in J} |y_1(t) - y_2(t)|.$$

Use the natural multi-dimensional generalization of the Cauchy Test to prove that this is a complete metric space. You need not *prove* the multi-dimensional generalization! Simply write down a careful statement of what you believe the generalization says, and apply this appropriately to deduce that B is a complete metric space.

(d)(5 points) Prove that for each continuous function y in B, the following function z is also in B,

$$z(x_1, \ldots, x_n) = g(x_1, \ldots, x_n, y(x_1, \ldots, x_n)).$$

Therefore the mapping $T(y) = z$ is a mapping from B into itself.

(e)(5 points) Prove that T is a u-contraction mapping. Use the Contraction Mapping Fixed Point Theorem to deduce that there exists a continuous function y in B such that $T(y) = y$. Deduce that f is 0 on the graph of y. (**Hint**: Use the mean value theorem.)

Problem 2(5 points) *The Implicit Function Theorem, Part II.* The notation is from Problem 1. Let $(x_1, \ldots, x_n, y)$ be a point in I such that $f(x_1, \ldots, x_n, y) = 0$. Prove that $y = y(x_1, \ldots, x_n)$. Therefore the points in I where f is 0 are exactly the points on the graph of $y(x_1, \ldots, x_n)$. (**Hint**: If $y \neq y(x_1, \ldots, x_n)$, use the mean value theorem to find a number y_1 between y and $y(x_1, \ldots, x_n)$ where the derivative $\frac{\partial g}{\partial y}(x_1, \ldots, x_n, y_1)$ gives a contradiction.)

Problem 3(10 points) Appendix A.1, Problem 1, p. 677.

Problem 4(5 points) Section 2.4, Problem 16, p. 67 (just draw a rough sketch; the definition of "Step" is on the inside front cover of the text).

Problem 5(5 points) Section 2.4, Problem 17, p. 67 (as above, just draw a rough sketch).