We're going to start.

We are going to start studying today, and for quite a while, the linear second-order differential equation with constant coefficients. In standard form, it looks like, there are various possible choices for the variable, unfortunately, so I hope it won't disturb you much if I use one rather than another. I'm going to write it this way in standard form. I'll use y as the dependent variable. Your book uses little p and little q. I'll probably switch to that by next time. But, for today, I'd like to use the most neutral letters I can find that won't interfere with anything else.

So, of course call the constant coefficients, respectively, capital A and capital B.

I'm going to assume for today that the right-hand side is zero. So, that means it's what we call homogeneous. The left-hand side must be in this form for it to be linear, it's second order because it involves a second derivative. These coefficients, A and B, are understood to be constant because, as I said, it has constant coefficients.

Of course, that's not the most general linear equation there could be. In general, it would be more general by making this a function of the dependent variable, x or t, whatever it's called.

Similarly, this could be a function of the dependent variable. Above all, the right-hand side can be a function of a variable rather than simply zero.

In that case the equation is called inhomogeneous.

But it has a different physical meaning, and therefore it's customary to study that after this.

You start with this. This is the case we start with, and then by the middle of next week we will be studying more general cases. But, it's a good idea to start here. Your book starts with, in general, some theory of a general linear equation of second-order, and even higher order.

I'm asking you to skip that for the time being.

We'll come back to it next Wednesday, it two lectures, in other words. I think it's much better and essential for your problems at for you to get some experience with a simple type of equation. And then, you'll understand the general theory, how it applies, a lot better, I think.

So, let's get experience here. The downside of that is that I'm going to have to assume a couple of things about the solution to this equation, how it looks; I don't think that will upset you too much.

So, what I'm going to assume, and we will justify it in a couple lectures, that the general solution, that is, the solution involving arbitrary constants, looks like this. y is equal-- The arbitrary constants occur in a certain special way.

There is c one y one plus c two y two.

So, these are two arbitrary constants corresponding to the fact that we are solving a second-order equation.

In general, the number of arbitrary constants in the solution is the same as the order of the equation because if it's a second-order equation because if it's a second-order equation, that means somehow or other, it may be concealed.

But you're going to have to integrate something twice to get the answer. And therefore, there should be two arbitrary constants.

That's very rough, but it sort of gives you the idea. Now, what are the y1 and y2?

Well, as you can see, if these are arbitrary constants, if I take c2 to be zero and c1 to be one, that means that y1 must be a solution to the equation, and similarly y2. So, where y1 and y2 are solutions. Now, what that shows you is that the task of solving this equation is reduced, in some sense, to finding just two solutions of it, somehow. All we have to do is find two solutions, and then we will have solved the equation because the general solution is made up in this way by multiplying those two solutions by arbitrary constants and adding them.

So, the problem is, where do we get that solutions from? But, first of all, or rather, second or third of all, the initial conditions enter into the, I haven't given you any initial conditions here, but if you have them, and I will illustrate them when I work problems, the initial conditions, well, the initial values are satisfied by choosing c1 and c2, are satisfied by choosing c1 and c2 properly. So, in other words, if you have an initial value problem to solve, that will be taken care of by the way those constants, c, enter into the solution. Okay, without further ado, there is a standard example, which I wish I had looked up in the physics syllabus for the first semester.

Did you study the spring-mass-dashpot system in 8.01? I'm embarrassed having to ask you. You did?

Raise your hands if you did. Okay, that means you all did.

Well, just let me draw an instant picture to remind you.

So, this is a two second review.

I don't know how they draw the picture.

Probably they don't draw picture at all.

They have some elaborate system here of the thing running back and forth. Well, in the math, we do that all virtually. So, here's my system.

That's a fixed thing. Here's a little spring.

And, there's a little car on the track here, I guess. So, there's the mass, some mass in the little car, and motion is damped by what's called a dashpot. A dashpot is the sort of thing, you see them in everyday life as door closers.

They're the thing up above that you never notice that prevent the door slamming shut. So, if you take one apart, it looks something like this. So, that's the dash pot.

It's a chamber with a piston. This is a piston moving in and out, and compressing the air, releasing it, is what damps the motion of the thing.

So, this is a dashpot, it's usually called.

And, here's our mass in that little truck.

And, here's the spring. And then, the equation which governs it is, let's call this x.

I'm already changing, going to change the dependent variable from y to x, but that's just for the sake of example, and because the track is horizontal, it seems more natural to call it x.

There's some equilibrium position somewhere, let's say, here. That's the position at which the mass wants to be, if the spring is not pulling on it or pushing on it, and the dashpot is happy.

I guess we'd better have a longer dashpot here.

So, this is the equilibrium position where nothing is happening. When you depart from that position, then the spring, if you go that way, the spring tries to pull the mass back.

If it goes on the site, the spring tries to push the mass away. The dashpot, meanwhile, is doing its thing. And so, the force on the, m x double prime. That's by Newton's law, the force, comes from where? Well, there's the spring pushing and pulling on it. That force is opposed.

If x gets to be beyond zero, then the spring tries to pull it back. If it gets to the left of zero, if x gets to be negative, that that spring force is pushing it this way, wants to get rid of the mass.

So, it should be minus kx, and this is from the spring, the fact that is proportional to the amount by which x varies.

So, that's called Hooke's Law. Never mind that.

This is a law. That's a law, Newton's law, okay, Newton, Hooke with an E, and the dashpot damping is proportional to the velocity. It's not doing anything if the mass is not moving, even if it's stretched way out of its equilibrium position. So, it resists the velocity.

If the thing is trying to go that way, the dashpot resists it. It's trying to go this way, the dashpot resists that, too.

It's always opposed to the velocity.

And so, this is a dash pot damping.

I don't know whose law this is. So, it's the force coming from the dashpot. And, when you write this out, the final result, therefore, is it's m x double prime plus c x prime, it's important to see where the various terms are, plus kx equals zero.

And now, that's still not in standard form. To put it in standard form, you must divide through by the mass.

And, it will now read like this, plus k divided by m times x equals zero. And, that's the equation governing the motion of the spring.

I'm doing this because your problem set, problems three and four, ask you to look at a little computer visual which illustrates a lot of things. And, I didn't see how it would make, you can do it without this interpretation of spring-mass-dashpot, -- -- but, I think thinking of it of these constants as, this is the damping constant, and this is the spring, the constant which represents the force being exerted by the spring, the spring constant, as it's called, makes it much more vivid. So, you will note is that those problems are labeled Friday or Monday.

Make it Friday. You can do them after today if you have the vaguest idea of what I'm talking about.

If not, go back and repeat 8.01.

So, all this was just an example, a typical model.

But, by far, the most important simple model. Okay, now what I'd like to talk about is the solution. What is it I have to do to solve the equation? So, to solve the equation that I outlined in orange on the board, the ODE, our task is to find two solutions.

Now, don't make it too trivial. There is a condition.

The solution should be independent.

All that means is that y2 should not be a constant multiple of y1. I mean, if you got y1, then two times y1 is not an acceptable value for this because, as you can see, you really only got one there.

You're not going to be able to make up a two parameter family.

So, the solutions have to be independent, which means, to repeat, that neither should be a constant multiple of the other. They should look different.

That's an adequate explanation. Okay, now, what's the basic method to finding those solutions?

Well, that's what we're going to use all term long, essentially, studying equations of this type, even systems of this type, with constant coefficients.

The basic method is to try y equals an exponential.

Now, the only way you can fiddle with an exponential is in the constant that you put up on top.

So, I'm going to try y equals e to the rt.

Notice you can't tell from that what I'm using as the independent variable. But, this tells you I'm using t. And, I'm switching back to using t as the dependent variable.

So, T is the independent variable.

Why do I do that? The answer is because somebody thought of doing it, probably Euler, and it's been a tradition that's handed down for the last 300 or 400 years. Some things we just know.

All right, so if I do that, as you learned from the exam, it's very easy to differentiate exponentials.

That's why people love them. It's also very easy to integrate exponentials. And, half of you integrated instead of differentiating. So, we will try this and see if we can pick r so that it's a solution.

Okay, well, I will plug in, then.

Substitute, in other words, and what do we get?

Well, for y double prime, I get r squared e to the rt.

That's y double prime because each time you differentiate it, you put an extra power of r out in front.

But otherwise, do nothing.

The next term will be r times, sorry, I forgot the constant.

Capital A times r e to the rt, and then there's the last term, B times y itself, which is B e to the rt.

And, that's supposed to be equal to zero.

So, I have to choose r so that this becomes equal to zero.

Now, you see, the e to the rt occurs as a factor in every term, and the e to the rt is never zero. And therefore, you can divide it out because it's always a positive number, regardless of the value of t. So, I can cancel out from each term. And, what I'm left with is the equation r squared plus ar plus b equals zero.

We are trying to find values of r that satisfy that equation. And that, dear hearts, is why you learn to solve quadratic equations in high school, in order that in this moment, you would be now ready to find how spring-mass systems behave when they are damped.

This is called the characteristic equation.

The characteristic equation of the ODE, or of the system of the spring mass system, which it's modeling, the characteristic equation of the system, okay?

Okay, now, we solve it, but now, from high school you know there are several cases. And, each of those cases corresponds to a different behavior.

And, the cases depend upon what the roots look like.

The possibilities are the roots could be real, and distinct. That's the easiest case to handle. The roots might be a pair of complex conjugate numbers. That's harder to handle, but we are ready to do it. And, the third case, which is the one most in your problem set is the most puzzling: when the roots are real, and equal.

And, I'm going to talk about those three cases in that order.

So, the first case is the roots are real and unequal.

If I tell you they are unequal, and I will put down real to make that clear. Well, that is by far the simplest case because immediately, one sees we have two roots. They are different, and therefore, we get our two solutions immediately. So, the solutions are, the general solution to the equation, I write down without further ado as y equals c1 e to the r1 t plus c2 e to the r2 t.

There's our solution. Now, because that was so easy, and we didn't have to do any work, I'd like to extend this case a little bit by using it as an example of how you put in the initial conditions, how to put in the c.

So, let me work a specific numerical example, since we are not going to try to do this theoretically until next Wednesday. Let's just do a numerical example. So, suppose I take the constants to be the damping constant to be a four, and the spring constant, I'll take the mass to be one, and the spring constant to be three.

So, there's more damping here, damping force here.

You can't really talk that way since the units are different.

But, this number is bigger than that one.

That seems clear, at any rate.

Okay, now, what was the characteristic equation?

Look, now watch. Please do what I do.

I've found in the past, even by the middle of the term, there are still students who feel that they must substitute y equals e to the rt, and go through that whole little derivation to find that you don't do that.

It's a waste of time. I did it that you might not ever have to do it again. Immediately write down the characteristic equation. That's not very hard.

r squared plus 4r plus three equals zero.

And, if you can write down its roots immediately, splendid.

But, let's not assume that level of competence.

So, it's r plus three times r plus one equals zero.

Okay, you factor it.

This being 18.03, a lot of the times the roots will be integers when they are not, God forbid, you will have to use the quadratic formula.

But here, the roots were integers.

It is, after all, only the first example.

So, the solution, the general solution is y equals c1 e to the negative, notice the root is minus three and minus one, minus 3t plus c2 e to the negative t.

Now, suppose it's an initial value problem.

So, I gave you an initial condition.

Suppose the initial conditions were that y of zero were one.

So, at the start, the mass has been moved over to the position, one, here. Well, we expected it, then, to start doing that. But, this is fairly heavily damped. This is heavily damped.

I'm going to assume that the mass starts at rest.

So, the spring is distended. The masses over here.

But, there's no motion at times zero this way or that way.

In other words, I'm not pushing it.

I'm just releasing it and letting it do its thing after that. Okay, so y prime of zero, I'll assume, is zero.

So, it starts at rest, but in the extended position, one unit to the right of the equilibrium position.

Now, all you have to do is use these two conditions.

Notice I have to have two conditions because there are two constants I have to find the value of.

All right, so, let's substitute, well, we're going to have to calculate the derivative.

So, why don't we do that right away?

So, this is minus three c1 e to the minus 3t minus c2 e to the negative t.

And now, if I substitute in at zero, when t equals zero, what do I get? Well, the first equation, the left says that y of zero should be one.

And, the right says this is one.

So, it's c1 plus c2.

That's the result of substituting t equals zero.

How about substituting?

What should I substitute in the second equation?

Well, y prime of zero is zero.

So, if the second equation, when I put in t equals zero, the left side is zero according to that initial value, and the right side is negative three c1 minus c2.

You see what you end up with, therefore, is a pair of simultaneous linear equations. And, this is why you learn to study linear set of pairs of simultaneous linear equations in high school. These are among the most important. Solving problems of this type are among the most important applications of that kind of algebra, and this kind of algebra.

All right, what's the answer finally?

Well, if I add the two of them, I get minus 2c1 equals one.

So, c1 is equal to minus one half.

And, if c1 is minus a half, then c2 is minus 3c1.

So, c2 is three halves.

The final question is, what does that look like as a solution? Well, in general, these combinations of two exponentials aren't very easy to plot by yourself. That's one of the reasons you are being given this little visual which plots them for you.

All you have to do is, as you'll see, set the damping constant, set the constants, set the initial conditions, and by magic, the curve appears on the screen.

And, if you change either of the constants, the curve will change nicely right along with it.

So, the solution is y equals minus one half e to the minus 3t plus three halves e to the negative t.

How does it look?

Well, I don't expect you to be able to plot that by yourself, but you can at least get started.

It does have to satisfy the initial conditions.

That means it should start at one, and its starting slope is zero. So, it starts like that.

These are both declining exponentials.

This declines very rapidly, this somewhat more slowly.

It does something like that. If this term were a lot, lot more negative, I mean, that's the way that particular solution looks. How might other solutions look?

I'll draw a few other possibilities.

If the initial term, if, for example, the initial slope were quite negative, well, that would have start like this.

Now, just your experience of physics, or of the real world suggests that if I give, if I start the thing at one, but give it a strongly negative push, it's going to go beyond the equilibrium position, and then come back again.

But, because the damping is big, it's not going to be able to get through that. The equilibrium position, a second time, is going to look something like that. Or, if I push it in that direction, the positive direction, that it starts off with a positive slope. But it loses its energy because the spring is pulling it. It comes and does something like that. So, in other words, it might go down. Cut across the equilibrium position, come back again, it do that?

No, that it cannot do. I was considering giving you a problem to prove that, but I got tired of making out the problems set, and decided I tortured you enough already, as you will see.

So, anyway, these are different possibilities for the way that can look. This case, where it just returns in the long run is called the over-damped case, over-damped. Now, there is another case where the thing oscillates back and forth.

We would expect to get that case if the damping is very little or nonexistent. Then, there's very little preventing the mass from doing that, although we do expect if there's any damping at all, we expect it ultimately to get nearer and nearer to the equilibrium position.

Mathematically, what does that correspond to?

Well, that's going to correspond to case two, where the roots are complex. The roots are complex, and this is why, let's call the roots, in that case we know that the roots are of the form a plus or minus bi. There are two roots, and they are a complex conjugate.

All right, let's take one of them.

What does a correspond to in terms of the exponential?

Well, remember, the function of the r was, it's this r when we tried our exponential solution.

So, what we formally, this means we get a complex solution. The complex solution y equals e to this, let's use one of them, let's say, (a plus bi) times t.

The question is, what do we do that? We are not really interested, I don't know what a complex solution to that thing means.

It doesn't have any meaning. What I want to know is how y behaves or how x behaves in that picture.

And, that better be a real function because otherwise I don't know what to do with it. So, we are looking for two real functions, the y1 and the y2. But, in fact, what we've got is one complex function.

All right, now, a theorem to the rescue: this, I'm not going to save for Wednesday because it's so simple. So, the theorem is that if you have a complex solution, u plus iv, so each of these is a function of time, u plus iv is the complex solution to a real differential equation with constant coefficients. Well, it doesn't have to have constant coefficients. It has to be linear.

Let me just write it out to y double prime plus A y prime plus B y equals zero.

Suppose you got a complex solution to that equation.

These are understood to be real numbers.

They are the damping constant and the spring constant.

Then, the conclusion is that u and v are real solutions.

In other words, having found a complex solution, all you have to do is take its real and imaginary parts, and voila, you've got your two solutions you were looking for for the original equation.

Now, that might seem like magic, but it's easy.

It's so easy it's the sort of theorem I could spend one minute proving for you now. What's the reason for it?

Well, the main thing I want you to get out of this argument is to see that it absolutely depends upon these coefficients being real. You have to have a real differential equation for this to be true.

Otherwise, it's certainly not. So, the proof is, what does it mean to be a solution?

It means when you plug in A (u plus iv) plus, prime, plus B times u plus iv, what am I supposed to get?

Zero. Well, now, separate these into the real and imaginary parts. What does it say?

It says u double prime plus A u prime plus B u, that's the real part of this expression when I expand it out. And, I've got an imaginary part, too, which all have the coefficient i.

So, from here, I get v double prime plus i times A v prime plus i times B v.

So, this is the imaginary part. Now, here I have something with a real part plus the imaginary part, i, times the imaginary part is zero. Well, the only way that can happen is if the real part is zero, and the imaginary part is separately zero. So, the conclusion is that therefore this part must be zero, and therefore this part must be zero because the two of them together make the complex number zero plus zero i. Now, what does it mean for the real part to be zero? It means that u is a solution.

This, the imaginary part zero means v is a solution, and therefore, just what I said.

u and v are solutions to the real equation.

Where did I use the fact that A and B were real numbers and not complex numbers? In knowing that this is the real part, I had to know that A was a real number.

If A were something like one plus i, I'd be screwed, I mean, because then I couldn't say that this was the real part anymore.

So, saying that's the real part, and this is the imaginary part, I was using the fact that these two numbers, constants, were real constants: very important.

So, what is the case two solution?

Well, what are the real and imaginary parts of (a plus b i) t? Well, y equals e to the at + ibt. Okay, you've had experience.

You know how to do this now. That's e to the at times, well, the real part is, well, let's write it this way.

The real part is e to the at times cosine b t.

Notice how the a and b enter into the expression. That's the real part.

And, the imaginary part is e to the at times the sine of bt.

And therefore, the solution, both of these must, therefore, be solutions to the equation.

And therefore, the general solution to the ODE is y equals, now, you've got to put in the arbitrary constants. It's a nice thing to do to factor out the e to the at.

It makes it look a little better.

And so, the constants are c1 cosine bt and c2 sine bt.

Yeah, but what does that look like?

Well, you know that too. This is an exponential, which controls the amplitude. But this guy, which is a combination of two sinusoidal oscillations with different amplitudes, but with the same frequency, the b's are the same in both of them, and therefore, this is, itself, a purely sinusoidal oscillation. So, in other words, I don't have room to write it, but it's equal to, you know. It's a good example of where you'd use that trigonometric identity I spent time on before the exam. Okay, let's work a quick example just to see how this works out.

Well, let's get rid of this. Okay, let's now make the damping, since this is showing oscillations, it must correspond to the case where the damping is less strong compared with the spring constant.

So, the theorem is that if you have a complex solution, u plus iv, so each of these is a function of time, u plus iv is the complex solution to a real differential equation with constant coefficients.

A stiff spring, one that pulls with hard force is going to make that thing go back and forth, particularly at the dipping is weak.

So, let's use almost the same equation as I've just concealed.

But, do you remember a used four here?

Okay, before we used three and we got the solution to look like that. Now, we will give it a little more energy by putting some moxie in the springs.

So now, the spring is pulling a little harder, bigger force, a stiffer spring.

Okay, the characteristic equation is now going to be r squared plus 4r plus five is equal to zero.

And therefore, if I solve for r, I'm not going to bother trying to factor this because I prepared for this lecture, and I know, quadratic formula time, minus four plus or minus the square root of b squared, 16, minus four times five, 16 minus 20 is negative four all over two. And therefore, that makes negative two plus or minus, this makes, simply, i. 2i divided by two, which is i. So, the exponential solution is e to the negative two, you don't have to write this in.

You can get the thing directly.

t, let's use the one with the plus sign, and that's going to give, as the real solutions, e to the negative two t times cosine t, and e to the negative 2t times the sine of t.

And therefore, the solution is going to be y equals e to the negative 2t times (c1 cosine t plus c2 sine t).

If you want to put initial conditions, you can put them in the same way I did them before. Suppose we use the same initial conditions: y of zero equals one, and y prime of zero equals one, equals zero, let's say, wait, blah, blah, blah, zero, yeah. Okay, I'd like to take time to actually do the calculation, but there's nothing new in it.

I'd have to take, calculate the derivative here, and then I would substitute in, solve equations, and when you do all that, just as before, the answer that you get is y equals e to the negative 2t, so, choose the constants c1 and c2 by solving linear equations, and the answer is cosine t, so, c1 turns out to be one, and c2 turns out to be two, I hope. Okay, I want to know, but what does that look like? Well, use that trigonometric identity. The e to the negative 2t is just a real factor which is going to reproduce itself. The question is, what is cosine t plus 2 sine t look like?

What's its amplitude as a pure oscillation?

It's the square root of one plus two squared.

Remember, it depends on looking at that little triangle, which is one, two, this is a different scale than that.

And here is the square root of five, right?

And, here is phi, the phase lag.

So, it's equal to the square root of five.

So, it's the square root of five times e to the negative 2t, and the stuff inside is the cosine of, the frequency is one. Circular frequency is one, so it's t minus phi, where phi is this angle.

How big is that, one and two?

Well, if this were the square root of three, which is a little less than two, it would be 60 infinity.

So, this must be 70 infinity. So, phi is 70 infinity plus or minus five, let's say. So, it looks like a slightly delayed cosine curve, but the amplitude is falling.

So, it has to start. So, if I draw it, here's one, here is, let's say, the square root of five up about here. Then, the square root of five times e to the negative 2t looks maybe something like this. So, that's square root of five e to the negative 2t.

This is cosine t, but shoved over by not quite pi over two. It starts at one, and with the slope zero. So, the solution starts like this. It has to be guided in its amplitude by this function out there, and in between it's the cosine curve. But it's moved over.

So, if this is pi over two, the first time it crosses, it's 72 infinity to the right of that. So, if this is pi over two, it's pi over two plus 70 infinity where it crosses.

So, it must be doing something like this.

And now, on the other side, it's got to stay within the same amplitude. So, it must be doing something like this. Okay, that gets us to, if this is the under-damped case, because if you're trying to do this with a swinging door, it means the door's going to be swinging back and forth. Or, our little mass now hidden, but you could see it behind that board, is going to be doing this. But, it never stops.

It never stops. It doesn't realize, but not in theoretical life. So, this is the under-damped.

All right, so it's like Goldilocks and the Three Bears.

That's too hot, and this is too cold.

What is the thing which is just right?

Well, that's the thing you're going to study on the problem set. So, just right is called critically damped. It's what people aim for in trying to damp motion that they don't want.

Now, what's critically damped? It must be the case just in between these two. Neither complex, nor the roots different. It's the case of two equal roots. So, r squared plus Ar plus B equals zero has two equal roots.

Now, that's a very special equation.

Suppose we call the root, since all of these, notice these roots in this physical case.

The roots always turn out to be negative numbers, or have a negative real part. I'm going to call the root a.

So, r equals negative a, the root.

a is understood to be a positive number.

I want that root to be really negative.

Then, the equation looks like, the characteristic equation is going to be r plus a, right, if the root is negative a, squared because it's a double root.

And, that means the equation is of the form r squared plus two times a r plus a squared equals zero.

In other words, the ODE looked like this.

The ODE looked like y double prime plus 2a y prime plus, in other words, the damping and the spring constant were related in this special wake, that for a given value of the spring constant, there was exactly one value of the damping which produced this in between case.

Now, what's the problem connected with it?

Well, the problem, unfortunately, is staring us in the face when we want to solve it.

The problem is that we have a solution, but it is y equals e to the minus at. I don't have another root to get another solution with. And, the question is, where do I get that other solution from?

Now, there are three ways to get it.

Well, there are four ways to get it.

You look it up in Euler. That's the fourth way.

That's the real way to do it. But, I've given you one way as problem number one on the problem set.

I've given you another way as problem number two on the problem set. And, the third way you will have to wait for about a week and a half.

And, I will give you a third way, too.

By that time, you won't want to see any more ways. But, I'd like to introduce you to the way on the problem set. And, it is this, that if you know one solution to an equation, which looks like a linear equation, in fact, the piece can be functions of t.

They don't have to be constant, so I'll use the books notation with p's and q's. y prime plus q y equals zero.

If you know one solution, there's an absolute, ironclad guarantee, if you'll know that it's true because I'm asking you to prove it for yourself.

There's another of the form, having this as a factor, one solution y one, let's call it, y equals y1 u is another solution.

And, you will be able to find u, I swear.

Now, let's, in the remaining couple of minutes carry that out just for this case because I want you to see how to arrange the work nicely. And, I want you to arrange your work when you do the problem sets in the same way.

So, the way to do it is, the solution we know is e to the minus at. So, we are going to look for a solution to this differential equation.

That's the differential equation.

And, the solution we are going to look for is of the form e to the negative at times u.

Now, you're going to have to make calculation like this several times in the course of the term.

Do it this way. y prime equals, differentiate, minus a e to the minus a t u plus e to the minus a t u prime.

And then, differentiate again.

The answer will be a squared.

You differentiate this: a squared e to the negative at u. I'll have to do this a little fast, but the next term will be, okay, minus, so this times u prime, and from this are you going to get another minus. So, combining what you get from here, and here, you're going to get minus 2a e to the minus a t u prime.

And then, there is a final term, which comes from this, e to the minus a t u double prime.

Two of these, because of a piece here and a piece here combine to make that. And now, to plug into the equation, you multiply this by one.

In other words, you don't do anything to it.

You multiply this line by 2a, and you multiply that line by a squared, and you add them.

On the left-hand side, I get zero.

What do I get on the right? Notice how I've arrange the work so it adds nicely. This has a squared times this, plus 2a times that, plus one times that makes zero.

2a times this plus one times this makes zero.

All that survives is e to the a t u double prime, and therefore, e to the minus a t u double prime is equal to zero.

So, please tell me, what's u double prime?

It's zero. So, please tell me, what's u? It's c1 t plus c2.

Now, that gives me a whole family of solutions. Just t would be enough because all I am doing is looking for one solution that's different from e to the minus a t.

And, that solution, therefore, is y equals e to the minus a t times t.

And, there's my second solution.

So, this is a solution of the critically damped case.

And, you are going to use it in three or four of the different problems on the problem set. But, I think you can deal with virtually the whole problem set, except for the last problem, now.