18.03 Recitation 23, May 4, 2010

Linear phase portraits

The matrices I want you to study all have the form $A = \begin{bmatrix} a & 2 \\ -2 & -1 \end{bmatrix}$.

1. Compute the trace, determinant, characteristic polynomial, and eigenvalues, in terms of a.

The trace is the sum of diagonals: $a-1$. The determinant is $-a+4$. the characteristic polynomial is $(a-\lambda)(-1-\lambda)+4 = \lambda^2 + (1-a)\lambda + 4 - a$. The eigenvalues are the roots of the characteristic polynomial, or $\frac{a-1\pm\sqrt{a^2+2a-15}}{2}$.

2. For these matrices, express the determinant as a function of the trace. Sketch the $(\text{tr } A, \det A)$ plane, along with the critical parabola $\det A = (\text{tr } A)^2/4$, and plot the curve representing the relationship you found for this family of matrices. On this curve, plot the points corresponding to the following values of a: $a = -6, -5, -2, 1, 2, 3, 4, 5$.

$\det A = 3 - \text{tr } A$. The points are: $(-7, 10)$, $(-6, 9)$, $(-3, 6)$, $(0, 3)$, $(1, 2)$, $(2, 1)$, $(3, 0)$, and $(4, -1)$. The line intersects the parabola at $(-6, 9)$ and $(2, 1)$, i.e., where $a = -5$ and $a = 3$, respectively.

3. Make a table showing for each a in this list (1) the eigenvalues; (2) information about the phase portrait derived from the eigenvalues (saddle, node, spiral) and the stability type (stable if all real parts are negative; unstable if at least one real part is positive; undesignated if neither); (3) further information beyond what the eigenvalues alone tell you: if a spiral, the direction (clockwise or counterclockwise) of motion; if the eigenvalues are repeated, whether the matrix is defective or complete.

a	eigenvalues	info	more info
-6	$-5,\ -2$	stable node	
-5	-3	defective stable node	
-2	$\frac{-3\pm\sqrt{-15}}{2}$	stable spiral	clockwise
1	$\pm\sqrt{-3}$	center	clockwise
2	$\frac{1\pm\sqrt{-7}}{2}$	unstable spiral	clockwise
3	1	defective unstable node	
4	$0,\ 3$	unstable comb	
5	$2\pm\sqrt{5}$	saddle	

The defective nodes are defective, because $A - \lambda I$ is nonzero.

4. Make sure you know how to find the general solution to $\dot{\mathbf{u}} = A\mathbf{u}$ for each of these cases. Special attention is required in the defective node case.

18.03 Differential Equations
Spring 2010