18.03 Recitation 22, April 29, 2010

Eigenvalues and Eigenvectors

1. We'll solve the system of equations $\begin{cases} \dot{x} &= -5x - 3y \\ \dot{y} &= 6x + 4y \end{cases}$

(a) Write down the matrix of coefficients, A, so that we are solving $\dot{\mathbf{u}} = A\mathbf{u}$. What is its trace? Its determinant? Its characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$? Relate the trace and determinant to the coefficients of $p_A(\lambda)$.

We write $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, so our equations become:

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} \\ &= \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &= A\mathbf{u}(t) \end{aligned}$$

The matrix A is then $\begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}$. Its trace is $-5 + 4 = -1$, its determinant is $(-5)(4) - (6)(-3) = -2$, and its characteristic polynomial is $(-\lambda - 5)(4 - \lambda) + 18 = \lambda^2 + \lambda - 2$. The trace is minus one times the linear term, and the determinant is the constant term.

(b) Find the eigenvalues and then for each eigenvalue find a nonzero eigenvector.

$\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$, so the eigenvalues are 1 and -2. $A - I = \begin{bmatrix} -6 & -3 \\ 6 & 3 \end{bmatrix}$, so $(A - I)v = 0$ has a solution $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. $A + 2I = \begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix}$, so $(A + 2I)v = 0$ has a solution $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(c) Draw the eigenlines and discuss the solutions whose trajectories live on each. Explain why each eigenline is made up of three distinct non-intersecting trajectories. Begin to construct a phase portrait by indicating the direction of time on portions of the eigenlines. Pick a nonzero point on an eigenline and write down all the solutions to $\dot{\mathbf{u}} = A\mathbf{u}$ whose trajectories pass through that point.

The eigenlines are the lines that contain the solutions that run through the origin with constant direction. The lines have the form $c\mathbf{v}$ for $c \in \mathbb{R}$, $\mathbf{v}$ an eigenvector. The solutions have the form $ce^{\lambda t}\mathbf{v}$ for $\mathbf{v}$ an eigenvector with eigenvalue λ. In our case, we have two lines: $y = -x$ and $y = -2x$, with trajectories $\mathbf{u}_1(t) = c_1 e^t \begin{bmatrix} 1 \\ -2 \end{bmatrix} = e^t \begin{bmatrix} c_1 \\ -2c_1 \end{bmatrix}$, and $\mathbf{u}_2(t) = c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{-2t} \begin{bmatrix} c_2 \\ -c_2 \end{bmatrix}$. The three nonintersecting trajectories arise because $e^{\lambda t}$ is always positive, and changing t does not switch between the parts of the line where c is positive, zero, or negative.

For each eigenvalue, the trajectory with positive c lies in the fourth quadrant, the trajectory with $c = 0$ lies at the origin, and the trajectory with negative c lies in the second quadrant.

(d) Now study the solution $\mathbf{u}(t)$ such that $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of a vector from the first eigenline and a vector from the second eigenline. Use this decomposition to express the solution, and sketch its trajectory. What is the general solution with this trajectory?

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so $a + b = 1$ and $-2a - b = 0$. Combining these, we see that $a = -1$ and $b = 2$. Then $\mathbf{u}(t) = ae^t \begin{bmatrix} 1 \\ -2 \end{bmatrix} + be^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -e^t + 2e^{-2t} \\ 2e^t - 2e^{-2t} \end{bmatrix}$.
The general solution can be found by substituting $t - \alpha$ for t.

(e) Fill out the phase portrait.

2. Same sequence of steps for $\begin{cases} \dot{x} = 4x + 3y \\ \dot{y} = -6x - 5y \end{cases}$

In this case, $A = \begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix}$. Its trace is $4 - 5 = -1$, its determinant is $(4)(-5) - (3)(-6) = -2$. Since this has the same trace and determinant as the previous problem, the characteristic polynomial is the same: $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$.

$A - I = \begin{bmatrix} 3 & 3 \\ -6 & -6 \end{bmatrix}$, so one eigenvector for the eigenvalue 1 is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $A + 2I = \begin{bmatrix} 6 & 3 \\ -6 & -3 \end{bmatrix}$, so one eigenvector for the eigenvalue -2 is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.
The eigenlines are $y = -x$ for the eigenvalue 1, and $y = -2x$ for the eigenvalue -2.
The trajectories have the form $c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

As before, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so one solution passing through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.
The phase portrait has the same eigenlines, but the flow is in the opposite direction.