18.03 Recitation 17, April 8, 2010

Convolution

Convolution product: $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$

Convention: We form the convolution product only of functions which vanish for $t < 0$. We may write $f(t)$, but we really mean $u(t)f(t)$.

Assertion: Suppose that $w(t)$ is the unit impulse response for the operator $p(D)$. Let $q(t)$ be a (perhaps generalized) function such that $q(t) = 0$ for $t < 0$. Then the solution to $p(D)x = q(t)$ with rest initial conditions is $w(t) * q(t)$.

1. (a) Compute $t * u(t)$. More generally, compute $q(t) * u(t)$ in terms of $q(t)$.

(b) Compute $u(t) * t$. More generally, compute $u(t) * q(t)$ in terms of $q(t)$.

Your answers should be related. What general property of the convolution product does this reflect?

2. What is the differential operator $p(D)$ whose unit impulse response is the unit step function $u(t)$?

In **1(b)** you have computed $u(t) * q(t)$. Is the Assertion true in this case?

3. (a) Assume that $f(t)$ is continuous at $t = a$. What meaning should we give to the product $f(t)\delta(t-a)$?

(b) Assume $f(t)$ is continuous, and $f(t)$ vanishes for $t < 0$. Explain why $f(t) * \delta(t-a) = f(t-a)$ for $a \geq 0$.

With $a = 0$, this shows that $\delta(t)$ serves as a "unit" for the convolution product.

4. (a) Verify that $u(t)\frac{1}{\omega_n}\sin(\omega_n t)$ is the unit impulse response for $D^2 + \omega_n^2 I$.

(b) Find the solution to $\ddot{x} + x = \sin t$ with initial condition $x(0) = \dot{x}(0) = 0$, using the ERF/resonance.

(c) Compute $\sin t * \sin t$ at $t = 2\pi n$, where n is a positive integer. (Reminder: $\sin^2 t = \frac{1-\cos(2t)}{2}$.)

By the Assertion, $\sin t * \sin t$ should be the solution found in **(b)**. Is the value at $t = 2\pi n$ correct?