Recitation 15, April 1, 2010

Fourier Series: Harmonic response

If $g(x)$ is a piecewise continuous periodic function and $2L$ is a period, then

$$g(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots + b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

The Fourier coefficients are defined as the numbers fitting into this expression. They can be calculated using the integral formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\,, \qquad b_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

$\ddot{x} + \omega_n^2 x = A\cos(\omega t)$ has solution $A\dfrac{\cos(\omega t)}{\omega_n^2 - \omega^2}$ and
$\ddot{x} + \omega_n^2 x = A\sin(\omega t)$ has solution $A\dfrac{\sin(\omega t)}{\omega_n^2 - \omega^2}$ as long as $\omega \neq \omega_n$.

1. Let $f(t)$ denote the even function $f(t)$ which is periodic of period 2π and such that $f(t) = |t|$ for $-\pi < t < \pi$. Graph $f(t)$. In lecture we found that the Fourier series of $f(t)$ is given by

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right)$$

Now we want to alter $f(t)$ to produce a function $g(t)$ whose graph is the same as that of $f(t)$ but is compressed (or expanded) horizontally so that the circular frequency is ω. What is the formula for $g(t)$ in terms of $f(t)$? Use the Fourier series for $f(t)$ and a substitution to find the Fourier series for the function $g(t)$.

2. Next drive the harmonic oscillator with the function $f(t)$ from **(1)**:
$\ddot{x} + \omega_n^2 x = f(t)$. Find a periodic solution, when one exists, as a Fourier series.

3. Now drive the harmonic oscillator with the function $g(t)$ from **(1)** of circular frequency ω: $\ddot{x} + \omega_n^2 x = g(t)$. Again, find a periodic solution, when one exists.

4. Suppose that ω is fixed, but we can vary ω_n. We may have a radio receiver, for example, and we want to pick up (amplify) radio signals at or near a certain circular frequency, so we set the capacitance so that the natural circular frequency of the circuit is ω_n. At what values of ω_n does the harmonic oscillator fail to have a periodic system response? (This is resonance.) Describe the system response when ω_n is just larger or just smaller than one of those values?

5. Are there frequencies at which there is more than one periodic solution?