Recitation 4, February 11, 2010

First order Linear ODEs: Integrating factors

Solution suggestions

1. Around here, the ocean experiences tides. About twice a day the ocean level rises and falls by several feet. This is why small boats are often tied up to floating docks.

A salt pond on Cape Cod is connected to the ocean by means of a narrow channel. This problem will explore how the water level in the pond varies.

In roughest terms, the water level in the bay increases, over a small time interval, by an amount which is proportional to (1) the difference between the ocean level and the bay level and (2) the length of the small time interval.

(a) Write $y(t)$ for the height of the ocean, measured against some zero mark, and $x(t)$ for the height of the bay, measured against the same mark. Set up the first order linear equation that describes this model. What is the "system" here? What part of the ODE represents it? What function is the "input signal"? What is the "output signal"?

We are given that over a small time interval, Δx is proportional to $(y - x)\Delta t$. Let us call the proportionality constant k. Then our differential equation is $x'(t) = k(y(t) - x(t))$. In standard form, this is

$$x' + kx = ky$$

The system is the bay (or salt pond), and its behavior is described by the left hand side. The input signal is the height of the ocean, and the output signal is the height of the bay.

(b) Assume that the tide is high exactly every 4π hours—not a bad approximation. Suppose that the ocean height is given by $y(t) = \cos(\omega t)$ (in meters and hours). What value does ω take?

When $t = 0$, $\cos(\omega 0) = \cos 0 = 1$ is a maximum, so the tide is high. The tide becomes high again when $\cos(\omega t) = 1$ again, i.e., when $\omega t = 2\pi$. At this time, $t = 4\pi$, so $\omega = 1/2$.

(c) Find a solution of your differential equation using integrating factors. You may find the following integral useful, a consequence, you will recall, of two integration by parts:

$$\int e^{kt} \cos(\omega t)\, dt = \frac{1}{k^2 + \omega^2} e^{kt}(k \cos(\omega t) + \omega \sin(\omega t)) + c.$$

We want a function u such that $ux' + ukx = \frac{d}{dt}(ux)$. Solving $u' = ku$ yields $u = Ce^{kt}$. Multiplying our differential equation by u yields $\frac{d}{dt}(ux) = uky$, or $x = \left(Ce^{kt}\right)^{-1} \int Ce^{kt} ky\,dt = e^{-kt} \int e^{kt} k \cos(t/2)dt$. Set $\omega = 1/2$ in the integral above, we then have

$$x(t) = \frac{2k\sin(t/2) + 4k^2 \cos(t/2)}{1 + 4k^2} + ce^{-kt}.$$

To check against the original equation, we differentiate: $x' = k\frac{\cos(t/2) - 2k\sin(t/2)}{1+4k^2} - cke^{-kt}$, so $x' + kx = k\frac{\cos(t/2) + 4k^2 \cos(t/2)}{1+4k^2} = ky$.

From the standpoint of a physical model of tides, we may assume that $c = 0$ if $k \neq 0$, since otherwise, the solution in the distant past or distant future would be unrealistically large. For the case $k = 0$ (no flow between the pond and ocean), our equation becomes $x' = 0$, so solutions are constant.

(d) Your solution probably had the form $a\cos(\omega t) + b\sin(\omega t)$ for constants a, b. Find it a second time by subsituting this expression into the differential equation and solving for a and b.

Let's assume $c = 0$ from above. Then $x = a\cos(t/2) + b\sin(t/2)$ and $x' = -a/2\sin(t/2) + b/2\cos(t/2)$. We find that $x' + kx = (ak + b/2)\cos(t/2) + (bk - a/2)\sin(t/2) = k\cos(t/2) = ky$. When $t = 0$, we have $ak + b/2 = k$, so $b = -2ka + 2k$. When $t = \pi$, we have $bk - a/2 = 0$, so $a = 2bk = -4k^2a + 4k^2$. Then $a = \frac{4k^2}{1+4k^2}$ and $b = \frac{2k}{1+4k^2}$.

2. In Recitation 2 you studied the direction field of the differential equation $y' = x - 2y$. Now find the general solution of this equation analytically. In Recitation 2 you found a straight line solution. Does it occur within your general solution? Plot it and several other solutions, using your expression for the general solution. Do you have a funnel? Find the particular solution with initial value $y(0) = 1$.

We use integrating factors: In standard form, $y' + 2y = x$, so $u = Ce^{2x}$. Then $y = u^{-1}\int ux\,dx = e^{-2x}\int xe^{2x}dx$. Integrating by parts yields $\int xe^{2x}dx = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$. Therefore, $y = x/2 - 1/4 + ce^{-2x}$.

The straight line solution occurs when $c = 0$. There is a funnel coming from the exponentially decaying term.

3. Recognize the left hand side as the derivative of a product in order to find the general solution of $x^2 y' + 2xy = \sin(2x)$.

We see that $(x^2 y)' = x^2 y' + 2xy$. Thus, $x^2 y = -\frac{1}{2}\cos(2x) + c$, and $y = cx^{-2} - \frac{\cos(2x)}{2x^2}$.

18.03 Differential Equations
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