18. Impulse and step responses

In real life, we often do not know the parameters of a system (e.g. the spring constant, the mass, and the damping constant, in a spring-mass-dashpot system) in advance. We may not even know the order of the system—there may be many interconnected springs (or diodes). (We will, however, suppose that all the systems we consider are linear and time independent, LTI.) Instead, we often learn about a system by watching how it responds to various input signals.

The simpler the signal, the clearer we should expect the signature of the system parameters to be, and the easier it should be to predict how the system will respond to other more complicated signals. To simplify things we will always begin the system from "rest."

In section we will study the response of a system from rest initial conditions to two standard and very simple signals: the unit impulse $\delta(t)$ and the unit step function $u(t)$.

The theory of the convolution integral, Section 19, gives a method of determining the response of a system to *any* input signal, given its unit impulse response.

18.1. **Impulse response.** In engineering one often tries to understand a system by studying its responses to known signals. Suppose for definiteness that the system is given by a first order left hand side $\dot{x}+p(t)x$. (The right hand side $q(t)$, isn't part of the "system"; it is the "input signal.") The variable x will be called the "system response," and in solving the ODE we are calculating that response. The analysis proceeds by starting "at rest," by which is meant $x(t) = 0$ for t less than the moment at which the signals occur. One then feeds the system various signals and watches the system response. In a certain sense the simplest signal it can receive is a delta function concentrated at some time t_0: $\delta(t - t_0)$. This signal is entirely concentrated at a single instant of time, but it has an effect nevertheless. In the case of a first order system, we have seen what that effect is, by thinking about what happens when I contribute a windfall to my bank account: for $t < t_0$, $x(t) = 0$; and for $t > t_0$, $x(t)$ is the solution to $\dot{x} + p(t)x = 0$ subject to the initial condition $x(t_0) = 1$. (Thus $x(t_0-) = 0$ and $x(t_0+) = 1$.) If $p(t) = a$ is constant, for example, this amounts to

$$x(t) = \begin{cases} 0 & \text{if } t < t_0 \\ e^{-a(t-t_0)} & \text{if } t > t_0. \end{cases}$$

This system response depends upon t_0, but if the system is LTI, as it is in this example, its dependence is very simple: The response to a unit impulse at $t = 0$ is called the **weight function** or **unit impulse response** of the system, or written $w(t)$. If the system is given by $\dot{x} + ax$, the weight function is given by

$$w(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-at} & \text{for } t > 0. \end{cases}$$

In terms of it, the response to a unit impulse at any time t_0 is

$$x(t) = w(t - t_0).$$

18.2. **Impulses in second order equations.** The word "impulse" comes from the interpretation of the delta function as a component of the driving term $q(t)$ in a second order system:

$$m\ddot{x} + b\dot{x} + cx = q(t). \tag{1}$$

In the mechanical interpretation of this equation, $q(t)$ is regarded as an external force acting on a spring-mass-dashpot system. Force affects acceleration, so the cumulative total of force, that is the time integral, affects velocity. If we have a very large force exerted over a very small time, the acceleration becomes very large for a short time, and the velocity increases sharply. In the limit we have an **impulse**, also known as a good swift kick. If $q(t) = a\delta(t - t_0)$, the system response is that the velocity $\dot{x}$ increases abruptly at $t = t_0$ by the quantity a/m. This produces a corner in the graph of x as a function of t, but not a break; the position does not change abruptly.

Thus the system response, $w(t)$, to a unit impulse at $t = 0$ is given for $t < 0$ by $w(t) = 0$, and for $t > 0$ by the solution to (1) subject to the initial condition $x(0) = 0$, $\dot{x}(0) = 1/m$.

For example, if the system is governed by the homogeneous LTI equation $\ddot{x} + 2\dot{x} + 5x = 0$, an independent set of real solutions is $\{e^{-t}\cos(2t), e^{-t}\sin(2t)\}$, and the solution to the initial value problem with $x(0) = 0$, $\dot{x}(0) = 1$, is $(1/2)e^{-t}\sin(2t)$. Thus

$$w(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ (1/2)e^{-t}\sin(2t) & \text{for} \quad t > 0. \end{cases}$$

This is illustrated in Figure 13. Note the aspect in this display: the vertical has been inflated by a factor of more than 10. In fact the slope $\dot{w}(0+)$ is 1.

The unit impulse response needs to be defined in two parts; it's zero for $t < 0$. This is a characteristic of *causal* systems: the impulse at

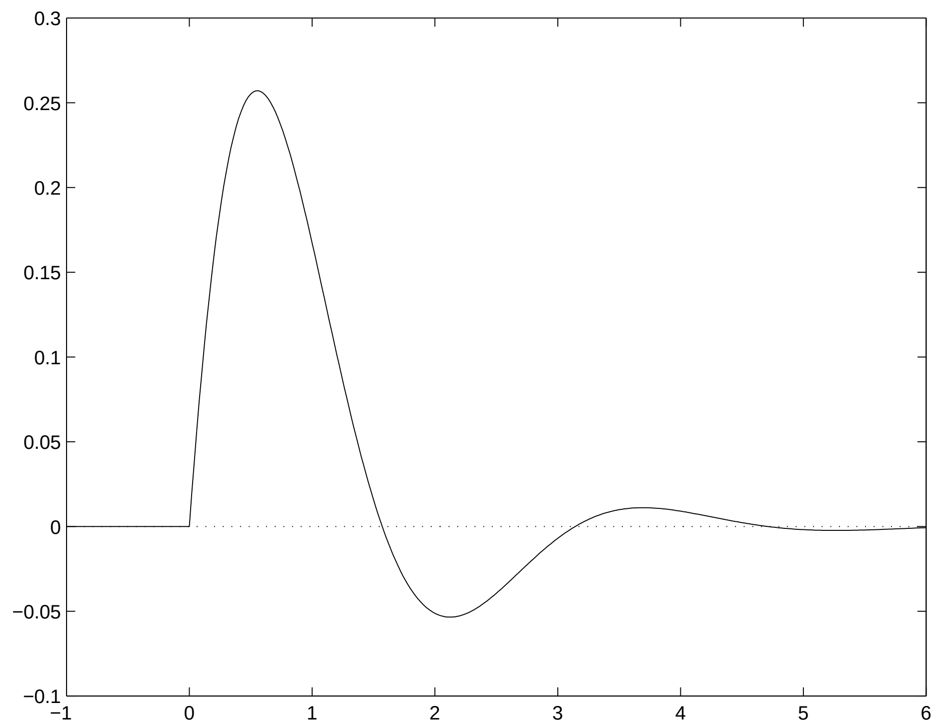

FIGURE 13. The weight function for $\ddot{x} + 2\dot{x} + 5x$

$t = 0$ has no effect on the system when $t < 0$. In a causal system the unit impulse response is always zero for negative time.

18.3. **Singularity matching.** Differentiation increases the order of singularity of a function. For example, the "ramp" function

$$\mathrm{ramp}(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ t & \text{for} \quad t > 0. \end{cases}$$

is not differentiable at $t = 0$ but it is continuous. Its derivative is the step function $u(t)$, which is not continuous at $t = 0$ but it is a genuine function; its singular part is zero. But *its* derivative is the delta function. (This can be made to continue; one can define an even more singular type of generalized function, of which $\delta'(t)$, often called a *doublet*, is an example, but we will not enter into this here.)

Suppose a function satisfies an ODE, say

$$m\ddot{x} + b\dot{x} + cx = q(t),$$

in which $q(t)$ may have a singular part. Whatever singularities x may have get accentuated by the process of differentiation, so the most singular part of $q(t)$ must match up with the most singular part of $m\ddot{x}$. This then forces x to be not too very singular; otherwise its second derivative would be more singular than $q(t)$.

To be more precise, if $q(t)$ is a generalized function in our sense, then its singular part must occur as the singular part of $m\ddot{x}$. The result is that $\dot{x}$ does not have a singular part, but does have discontinuities at the locations at which $q(t)$ has delta components. Similarly, x is continuous, but has jumps in its derivative at those locations. This makes physical sense: a second order system response to a generalized function is continuous but shows sudden jumps in velocity where the signal exhibits impulses.

This analysis is quantitative. If for example $q(t) = 3\delta(t) + 6t$, $m\ddot{x}$ has singular part $3\delta(t)$, so $\ddot{x}$ has singular part $(3/m)\delta(t)$. Thus $\dot{x}$ is continuous except at $t = 0$ where it has a jump in value of $3/m$; and x is differentiable except at $t = 0$, where its derivative jumps by $3/m$ in value.

In a first order system, say $m\dot{x} + kx = q(t)$, the singular part of $m\dot{x}$ is the singular part of $q(t)$, so x is continuous except at those places. If for example $q(t) = 3\delta(t) + 6t$, $\dot{x}$ has singular part $(3/m)\delta(t)$, so x jumps in value by $3/m$ at $t = 0$.

This line of reasoning is called “singularity matching.”

18.4. **Step response.** This is the response of a system at rest to a constant input signal being turned on at $t = 0$. I will write $w_1(t)$ for this system response. If the system is represented by the LTI operator $p(D)$, then $w_1(t)$ is the solution to $p(D)x = u(t)$ with rest initial conditions, where $u(t)$ is the unit step function.

The unit step response can be related to the unit impulse response using the following observation: The time invariance of $p(D)$ is equivalent to the fact that as operators

$$p(D)D = Dp(D).$$

We can see this directly:

$$(a_nD^n + \cdots + a_0I)D = a_nD^{n+1} + \cdots + a_0D = D(a_nD^n + \cdots + a_0I)\,.$$

Using this we can differentiate the equation $p(D)w_1 = 1$ to find that $p(D)(Dw_1) = \delta(t)$, with rest initial conditions. That is to say, $\dot{w}_1(t) = w(t)$, or:

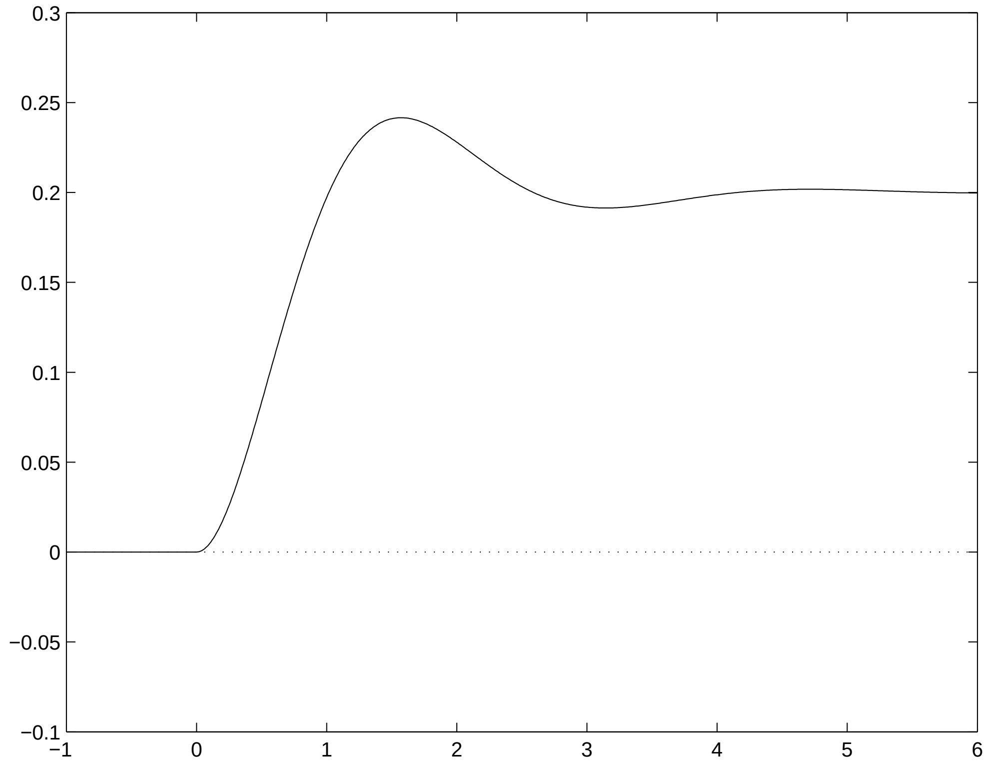

FIGURE 14. The unit step response for $\ddot{x} + 2\dot{x} + 5x$

The derivative of the unit step response is the unit impulse response.

If we return to the system represented by $\ddot{x} + 2\dot{x} + 5x$ considered above, a particular solution to $\ddot{x} + 2\dot{x} + 5x = 1$ is given by $x = 1/5$, so the general solution is $x = (1/5) + e^{-t}(a\cos(2t) + b\sin)2t))$. Setting $x(0) = 0$ and $\dot{x}(0) = 0$ leads to

$$w_1(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ (1/5) - (e^{-t}/10)(2\cos(2t) + \sin(2t)) & \text{for} \quad t > 0 \end{cases}$$

as illustrated in Figure 14. You can check that the derivative of this function is $w(t)$ as calculated above. In this example the unit impulse response is a simpler function than the unit step response, and this is generally the case.