18.03 Class 12, March 1, 2010

Good Vibrations

[0] Summary table [on the blackboard]
[1] Overdamping
[2] Underdamping
[3] Real solution theorem
[4] Natural and damped circular frequencies
[5] Critical damping
[6] Transience
[7] Root diagram

[0] Summary table of unforced system responses.

Name*	m,b,k relation	Char. roots	Exp. sol's	Basic real solns
Overdamped	b^2/4m > k	Two diff. real roots, r1, r2	e^{r1 t}, e^{r2 t}	same
Critically damped	b^2/4m = k	Repeated root r = - b/2m	e^{rt}	e^{rt}, te^{rt}
Underdamped	b^2/4m < k	-b/2m +- iw	e^{r1 t}, e^{r2 t}	e^{-bt/2m}cos(w t) e^{-bt/2m}sin(w t)

(w = omega_d = sqrt{ (k/m) - (b/2m)^2 })

* The names here are appropriate under the assumption that b and k are
both non-negative. The rest of the table makes sense in general, but it
doesn't have a good interpretation in terms of a mechanical system.

We are studying equations of the form

      mx" + bx' + kx = 0   (*)

which model a mass, dashpot, spring system without external forcing term.
Homogeneous constant coefficient linear second order.

We found that (*) has an exponential solution e^{rt} exactly when
r is a root of the "characteristic polynomial" p(s) = ms^2 + bs + k .
These are called the MODES of the system.

[1] Example. [Overdamped: Distinct real roots] x" + 5x' + 4x = 0 .
We did this on Friday:
The characteristic polynomial s^2 + 5s + 4 factors as (s + 1)(s + 4)
so the roots are r = -1 and r = -4 . The corresponding exponential
solutions are e^{-t} and e^{-4t} . These are called *modes* of the
system. They represent pure states. The general solution is mixture of
the two states, x = c_1 e^{-t} + c_2 e^{-4t} .

But where's the vibration?

[2] Example. [Underdamped: Nonreal roots] Make the spring a little stronger
and the dashpot a little weaker: x" + 4x' + 5x = 0

A good way to find the roots of a quadratic polynomial is to complete the
square:

    s^2 + 4s + 5 = (s + 2)^2 + 1 .

Setting this equal to zero, (s + 2)^2 = -1 or   s + 2 = +- i
or s = -2 +- i . So our exponential solutions are

    e^{(-2+i)t} , e^{(-2-i)t}

The general solution is a linear combination of these two "basic"
solutions. But I guess we were expecting REAL valued solutions.

For this we have:

[3] [Slide] Real Solution Theorem:
If z is a complex-valued solution to mz" + bz' + kz = 0 ,
where m , b , and k are real, then the real and imaginary parts of z
are also solutions.

Proof: Write z = u + iv   and build the table.

    k] z   = u + iv
    b] z' = u' + iv'
    m] z" = u" + iv"
        _____

         0 = (mu" + bu' + ku) + i(mv" + bv' + kv)

Both things in parentheses are real, so the only way this can happen is for
both of them to be zero.

In our situation,

e^{(-2+i)t}   has real part   e^{-2t} cos(t)
              and imaginary part   e^{-2t} sin(t)

so we have those two solutions.

If I had chosen the other exponential solution,

e^{(-2-i)t}   has real part   e^{-2t} cos(-t) = e^{-2t} cos(t)
              and imaginary part   e^{-2t} sin(-t) = - e^{-2t} sin(t)

(This also follows from the fact that the roots are complex conjugates of each
other and hence the two exponential functions are too.)

So you get the same (up to sign) basic real solutions using either solution.

Taking linear combinations, we get the
general solution

x = e^{-2t} (a cos(t) + b sin(t))

           or A e^{-2t} cos(t - phi)

This is a "damped sinusoid," with "circular pseudo-frequency" 1.

I demonstrated equations like this using the Mathlet Damped Vibration.

In this underdamped case, you should go straight to the answer and
not pass through the complex exponential, which was just a convenient
way to find the answer. If the roots are a +- omega i , the solutions
are e^{at} cos(omega t) and e^{at} sin(omega t) .

[4] Let's get formulas for the roots. The quadratic formula tells you,
or complete the square:

    0 = s^2 + (b/m) s + k/m = (s + (b/2m))^2 - ((b/2m)^2 - k/m)

So   r = -b/2m +- sqrt{(b/2m)^2 - k/m} .

Note that sqrt(-d) = i sqrt(d) , so if (b/2m)^2 < k/m then the roots are

    r = -b/2m +- omega_d i ,   omega_d = sqrt(k/m-(b/2m)^2)

It is called the "damped circular frequency" of the system represented by
mx" + bx' + kx .

Example. [Undamped, special case of Underdamped]
mx" + kx = 0 , k/m > 0 : no damping : "Harmonic Oscillator."

Roots are +- omega_n i , omega_n = sqrt{k/m} so we have exponential solutions
e^{i omega t} and e^{-i omega t} with real and imaginary parts
cos(omega_n t) and sin(omega_n t) . This recovers our earlier calculation.

Even if b is not zero, sqrt{k/m} is denoted omega_n and called the
"natural circular frequency of the system.

Question 12.1. As b increases from b = 0 (while m and k remain fixed)
what happens to the period of oscillation of nonzero solutions of
mx" + bx' + kx = 0 ?

1. It stays the same

2. It gets shorter

3. It gets longer

4. It depends on the other parameters

5. It depends on the initial conditions

Blank. Don't know

Ans: The quantity inside the square root decreases, so the circular
frequency decreases and the period increases. As b increases towards
critical damping, the period goes from 2pi/omega_n to infinity.

[5] In between overdamped and underdamped there is a marginal case, when
the roots are equal.

Example. [Critically damped]   x" + 4x' + 4 = 0.
Then p(s) = (s+2)^2 has r = -2 as a repeated root. The only
exponential solution is e^{-2t} . Another solution, not a constant
multiple of e^{-2t}$, is given by te^{-t} . I will not check this for
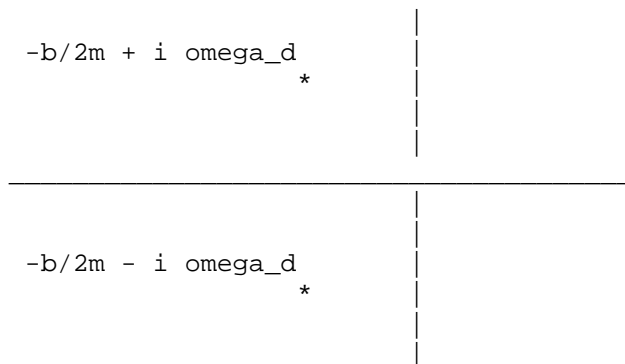you, you know how to do it: plug in and use the product rule.

So the general solution is   e^{-2t} (a+ct) in this case.

[6] Transience: If m, b, and k are all positive, then every solution
dies away for large t.

Look at the roots!   r = -b/2m +- sqrt{(b/2m)^2 - k/m} .

In the underdamped case, the amplitude is e^{-bt/2m} , which dies off.
In the overdamped case, the square root is smaller than b/2m , so
both roots are negative, and all solutions die off.
In the critically damped case, the root is negative so (even though you
get to multiply by t) all solutions die off.

[7] In the complex plane, the roots look like this:

                              |
        -b/2m + i omega_d |
                       * |
                              |
                              |
                              |
    _____
                              |
                              |
        -b/2m - i omega_d |
                       * |
                              |
                              |

I used the Damped Vibrations Mathlet to illustrate that as the
roots move to the left, the solutions decay faster; when they move
onto the imaginary axis, there is no damping. If we could envision
anti-damping, then we really get into Beach Boy territory and the solutions
grow exponentially in amplitude.

If the roots are pushed towards the real axis (by increasing the spring
constant, for example, then the frequency decreases; the period increases.
This can be a bit hard to see on the applet, but we did see it analytically.

When the two roots coalesce, we have critical damping, and no more vibration. In fact no solution crosses the t axis more than once. This behavior continues when the roots separate on the real axis, in the overdamped regime.

Muddy cards.

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