18.03 Class 7, Feb 17, 2010

Exponential and Sinusoidal input and output

[1] Sinusoidal functions
[2] Trig sum formula
[3] Integration of complex valued functions
[4] Linear equations with sinusoidal input signal
[5] Complex replacement

Euler: Re e^{(a+bi)t} = e^{at} cos(bt)
Im e^{(a+bi)t} = e^{at} sin(bt)

[1] Sinusoids

A "sinusoidal function" f(t) is one whose graph is a (co)sine wave.

I drew a large general sinusoidal function, f(t) .

I drew the graph of cos(theta) ; this is our model example of a sinusoid.

A sinusoidal function is entirely determined by just three measurements, or parameters, which determine it in terms of cos(theta) .

A = Amplitude = height of its maxima = depth of its minima

P = Period = elapsed time till it repeats
(or, in spatial terms, lambda = wavelength = the distance between repeats)

t_0 = Time lag = time of first maximum

f(t) can be written in terms of cosine. Clearly, f(t) = A cos(theta).
To work out how, express theta as a function of t . I started drew
a t-axis horizontally and a theta-axis vertically.
When t = t_0 , theta = 0 . When t = t_0 + P , theta = 2pi .
I marked these data points on all three graphs.

The graph of theta as a function of t is a straight line; otherwise the cosine would get distorted, bunched up. So:

theta = (2pi/P) (t - t_0)

and so f(t) = A cos((2pi/P) (t - t_0))

The frequency is nu = 1/P , measured in "cycles/unit time." More useful is:
the circular frequency omega = 2pi/P = omega , measured in radians/unit time.

So theta = omega (t - t_0)

The "phase lag" is

phi = omega t_0 = (2 pi/P) t_0 .

It measures the radian measure corresponding to t = 0 .
In terms of omega and phi ,

f(t) = A cos(omega t - phi)

For example,

sin(omega t) = cos(omega t - pi/2) .

-- the sine lags one quarter cycle behind the cosine.

Question 7.1. A graph of a sinusoidal function is displayed.
The problem is to express it in the "standard form" above.

1. 2 cos(4pi t + pi/4)
2. 2 cos((pi/4)t + pi/4)
3. 2 cos(4pi t - pi/4)
4. 2 cos((pi/4)t - pi/4)
5. 2 cos(4t+1)
6. 2 cos(4t-1)

P = 8, t_0 = -1 , A = 2 :

f(t) = 2 cos((2pi/8) (t + 1)) = 2 cos((pi/4) t + pi/4)

omega = pi/4 , phi = - pi/4 .

Ans: 2.

[2] Trig sum: a cos(omega t) + b sin(omega t)

I showed the Trig Id applet. This sum seems always to be another sinusoidal function! How can we find its "standard form" A cos(omega t - phi) ?

Recall the cosine difference identity:

cos(theta - phi) = cos(phi) cos(theta) + sin(phi) sin(theta)

cos(omega t - phi) = cos(phi) cos(omega t) + sin(phi) sin(omega t)

Now construct a right triangle with hypotenuse the segment in the plane joining (0,0) to (a,b) . If A is the hypotenuse and phi the angle at the origin, then

A = sqrt(a^2 + b^2)
a = A cos(phi)
b = A sin(phi)

and

A cos(omega t - phi) = a cos(omega t) + b sin(omega t)

Question 7.2. What are the amplitude, circular frequency, and phase lag:
A , omega , and phi in A cos(omega t - phi) , for the sinusoid

cos(omega t) + sqrt(3) sin(omega t)

1. 2 cos(\omega t-\frac{\pi}{4})
2. sqrt(3) cos(omega (t - pi/3))
3. 2 cos(omega(t- pi/3))
4. 2 cos(omega t - pi/3)
5. sqrt(3) cos(omega t - pi/3)
6. sqrt(3) cos(omega t - pi/4)

Blank. Don't know.

Ans: A = 2, phi = pi/3 : 4.

[3] Integration

Remember how to integrate e^{2t} cos(t) ?

Use parts twice. Or:

Differentiating a complex valued function is done separately on the real and imaginary parts. Same for integrating.

e^{2t} cos(t) = Re e^{(2+i)t}   so

int e^{2t} cos(t) dt = Re int e^{(2+i)t} dt

and we can integrate exponentials because we know how to differentiate them! -

int e^{(2+i)t} dt = (1/(2+i)) e^{(2+i)t} + c

We need the real part.

Expand everything out: 1/(2+i) = (2-i)/5

e^{(2+i)t} = e^{2t} (cos(t) + i sin(t))

so the real part of the product is

(1/5) e^{2t} (2 cos(t) + sin(t)) + c

More direct than the high school method!

[4] Linear constant coefficient ODEs with exponential input signal

Let's try   x' + 2x = 4 e^{3t}

We could use our integrating factor, but instead
let's use the method of "optimism," or the inspired guess. The inspiration here is based on the fact that differentiation reproduces exponentials:

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d
-- e^{rt} = r e^{rt}
dt
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Since the right hand side is an exponential, maybe the output signal x will be too: TRY x = A e^{3t} . This is not going to be the general solution, so I'll write x_p for it. I don't know what A is yet, but:

$$2 x_p = 2 A e^{3t}$$

$$x_p' = A 3 e^{3t}$$

$$4 e^{3t} = A (3+2) e^{3t}$$

which is OK as long as $A = 4/5$: $x_p = (4/5) e^{3t}$ is one solution.
The general solution is this plus a transient:

$$x = (4/5) e^{3t} + c e^{-2t} .$$

[6] Replacing sinusoidal signals with exponential ones

Let'e go back to the original ODE

$$x' + 2x = 2 \cos(t)$$

This equation is the real part of a complex valued ODE:

$$z' + 2z = 2 e^{it}$$

This is a different ODE, and I use a different variable name, $z(t)$.

We just saw how to get an exponential solution: $z_p = A e^{it}$

$$2 zp = 2 A e^{it}$$

$$zp' = i A e^{it}$$

$$2 e^{it} = A (2+i) e^{it}$$

so $z_p = 2/(i+2) e^{it}$

To get a solution to the original equation we should take the real part
of this! Expand each factor in real and imaginary parts:

$$z_p = (2(2-i)/5) (\cos(t) + i \sin(t))$$

$$x_p = Re(z_p) = (4/5) \cos(t) + (2/5) \sin(t)$$

This is the only sinusoidal solution. To get the general solution we add
a transient:

$$x = x_p + c x_h$$

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