

18.03 Class 6, Feb 16, 2010

Complex exponential

- [1] Complex roots
- [2] Complex exponential, Euler's formula
- [3] Euler's formula continued

[1] More practice with complex numbers:

Magnitudes Multiply:  $|zw| = |z||w|$   
Arguments Add:  $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$

so:  $|z^n| = |z|^n$  and  $\text{Arg}(z^n) = n \text{Arg}(z)$

For example:

Let's take some powers of  $z = 1+i$ .

$ z  = \sqrt{2}$	$\text{Arg}(z) = \pi/4$
$ z^2  = 2$	$\text{Arg}(z^2) = \pi/2$
$ z^3  = 2\sqrt{2}$	$\text{Arg}(z^3) = 3\pi/4$
$ z^4  = 4$	$\text{Arg}(z^4) = \pi$

Notice that these numbers march out along a spiral. This continues for all powers of  $1+i$ , even negative ones.

How about fractional powers? i.e. roots: of unity, first.

What are the cube roots of 1? Well 1 is one. If  $z$  is one, then  $|z|^3 = |z^3| = 1$ , so it lies on the unit circle.

The argument has to be so that 3 times it is zero - or  $2\pi$ , or  $4\pi$ , or ....

Thus the argument is

0, giving 1, or  
 $2\pi/3$ , giving  $(-1+\sqrt{3}i)/2$  or  
 $4\pi/3$ , giving  $(-1-\sqrt{3}i)/2$ .

The  $n$ th roots of unity divide the unit circle into  $n$  equal pieces.

How about cube roots of  $-8$ . If  $z^3 = -8$ , then  $|z|^3 = |z^3| = 8$ .  $|z|$  is a positive real number so  $|z| = 2$ : all cube roots of  $-8$  lie on the circle with center 0 and radius 2.

The argument of  $-8$  is  $\pi$ , giving a cube root with argument  $\pi/3$   
or  $3\pi$  giving a cube root with argument  $\pi$   
or  $5\pi$  giving a cube root with argument  $5\pi/3$

These again divide the circle evenly. The second is  $-2$ , and the others are  $1+\sqrt{3}i$  and  $1-\sqrt{3}i$ .

I gave a little demo with the Complex Roots applet.

We have been studying roots of equations like  $z^n - a = 0$ .

A "polynomial" is a function of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0, \quad a_n \neq 0$$

The "degree" is the top nonzero coefficient  $a_n$ .

Fundamental theorem of algebra (Gauss,...): Any polynomial of degree  $> 0$  has a complex root.

In fact any polynomial factors as

$$f(z) = a_n (z - r_1) (z - r_2) \dots (z - r_n)$$

It has  $n$  complex roots (some of which may coincide).

## [2] Complex exponential

The complex valued function  $z(t) = a(t) + i b(t)$  parametrizes a directed curve in the plane. For example

$z = t + i$  parametrizes a line running horizontally through  $i$ , directed towards the right.

The derivative is computed for each component, and gives you the velocity vector. Here this is  $1$ : horizontal.

Here's an ODE we can try to solve:  $z' = iz$ ,  $z(0) = 1$ . (\*\*)

In lecture 1 we saw that  $e^{kt}$  is the solution to  $x' = kx$ ,  $x(0) = 1$ . So we will write the solution to (\*\*) as  $z = e^{it}$ .

On the other hand we found out that multiplication by  $i$  is rotation by 90 degrees; so the solution is a curve such that the velocity vector is always perpendicular to the radius vector. This is a circle, and if we add the initial condition it is the unit circle, traversed counterclockwise with speed 1:

$$z = \cos t + i \sin t$$

To check, compute

$$z' = -\sin t + i \cos t$$

$$iz = i \cos t - \sin t$$

and they agree. Thus:

$$e^{it} = \cos t + i \sin t. \quad \text{"Euler's formula."}$$

This function parametrizes the unit circle, directed counterclockwise. This is the *definition* of  $e^{it}$ .

In fact, for any complex number  $a+bi$  you can compute that the solution to  $z' = (a+bi)z$ ,  $z(0) = 1$ , is

$$e^{(a+bi)t} = e^{at} (\cos(bt) + i \sin(bt))$$

That is, the magnitude of  $e^{\{(a+bi)t\}}$  is  $e^{\{at\}}$   
 and the argument of  $e^{\{(a+bi)t\}}$  is  $bt$

When  $a > 0$  and  $b > 0$ , we have a spiral moving counterclockwise away from the origin.

Q6.1 The curve in the complex plane traced out by  $e^{\{(1+2\pi i)t\}}$  most closely resembles which of the following?

- (1) A straight ray along the positive real axis
- (2) A circle of radius  $e$  and center at the origin.
- (3) A circle of radius 1 and center at the origin.
- (4) A spiral moving inwards and counterclockwise
- (5) A spiral moving outwards and counterclockwise
- (6) A spiral moving inwards and clockwise
- (7) A spiral moving outwards and clockwise

(Blank) Don't know.

Ans: (5).

[3] This definition (\*) satisfies the expected exponential rule:

$$e^{\{(z+w)t\}} = e^{\{wt\}} e^{\{zt\}}$$

You can see this using the usual rule for real exponentials together with the angle addition formulas, or by using the uniqueness theorem for solutions to ODEs. See the Supplementary Notes.

General fact about complex numbers:

$$\frac{z + \bar{z}}{2} = \text{Re}(z) \qquad \frac{z - \bar{z}}{2i} = \text{Im}(z)$$

Proof by diagram.

Apply this to  $z = e^{\{it\}}$ . I will need to know what  $\overline{e^{\{it\}}}$  is. Since complex conjugation preserves modulus and reverses angle,

$$\overline{e^{\{it\}}} = e^{\{-it\}} .$$

From Euler's formula (\*\*) and the "general fact" we find

$$\cos(t) = \frac{e^{\{it\}} + e^{\{-it\}}}{2} \qquad \sin(t) = \frac{e^{\{it\}} - e^{\{-it\}}}{2i}$$

Sometimes these are also called Euler's Formulas.

Anything you want to know about sines and cosines can be obtained from properties of the (complex) exponential function.

Neat examples:  $e^{\pi i} = -1$   
 $e^{2k \pi i} = 1$  for any whole number  $k$   
 $e^{\pi i / 2} = i$   
 $e^{(k/n) 2 \pi i} = \text{nth roots of unity.}$

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