

## Solutions of Spring 2008 Final Exam

1. (a) The isocline for slope 0 is the pair of straight lines  $y = \pm x$ . The direction field along these lines is flat.

The isocline for slope 2 is the hyperbola on the left and right of the straight lines. The direction field along this hyperbola has slope 2.

The isocline for slope  $-2$  is the hyperbola above and below the straight lines. The direction field along this hyperbola has slope  $-2$ .

- (b) The sketch should have the following features:

The curve passes through  $(-2, 0)$ . The slope at  $(-2, 0)$  is  $(-2)^2 - (0)^2 = 4$ .

Going backward from  $(-2, 0)$ , the curve goes down ( $dy/dx > 0$ ), crosses the left branch of the hyperbola  $x^2 - y^2 = 2$  with slope 2, and gets closer and closer to the line  $y = x$  but never touches it.

Going forward from  $(-2, 0)$ , the curve first goes up, crosses the left branch of the hyperbola  $x^2 - y^2 = 2$  with slope 2, and becomes flat when it intersects with  $y = -x$ . Then the curve goes down and stays between  $y = -x$  and the upper branch of the hyperbola  $x^2 - y^2 = -2$ , until it becomes flat as it crosses  $y = x$ . Finally, the curve goes up again and stays between  $y = x$  and the right branch of the hyperbola  $x^2 - y^2 = 2$  until it leaves the box.

- (c)  $f(100) \approx 100$
- (d) It follows from the picture in (b) that  $f(x)$  reaches a local maximum on the line  $y = -x$ . Therefore  $f(a) = -a$ .
- (e) Since we know  $f(-2) = 0$ , to estimate  $f(-1)$  with two steps, the step size is 0.5. At each step, we calculate

$$x_n = x_{n-1} + 0.5, \quad y_n = y_{n-1} + 0.5(x_{n-1}^2 - y_{n-1}^2)$$

The calculation is displayed in the following table.

$n$	$x_n$	$y_n$	$0.5(x_n^2 - y_n^2)$
0	-2	0	2
1	-1.5	2	-0.875
2	-1	1.125	

The estimate of  $f(-1)$  is  $y_2 = 1.125$ .

2. (a) The equation is  $\dot{x} = x(x-1)(x-2)$ . The phase line has three equilibria  $x = 0, 1, 2$ .  
 For  $x < 0$ , the arrow points down.  
 For  $0 < x < 1$ , the arrow points up.  
 For  $1 < x < 2$ , the arrow points down.  
 For  $x > 2$ , the arrow points up.
- (b) The horizontal axis is  $t$  and the vertical axis is  $x$ . There are three constant solutions  $x(t) \equiv 0, 1, 2$ . Their graphs are horizontal.  
 Below  $x = 0$ , all solutions are decreasing and they tend to  $-\infty$ .

Between  $x = 0$  and  $x = 1$ , all solutions are increasing and they approach  $x = 1$ .  
 Between  $x = 1$  and  $x = 2$ , all solutions are decreasing and they approach  $x = 1$ .  
 Above  $x = 2$ , all solutions are increasing and they tend to  $+\infty$ .

- (c) A point of inflection  $(a, x(a))$  is where  $\ddot{x}$  changes sign. In particular,  $\ddot{x}(a)$  must be zero. Differentiating the given equation with respect to  $t$ , we have

$$\ddot{x} = 2\dot{x} - 6x\dot{x} + 3x^2\dot{x} = \dot{x}(2 - 6x + 3x^2)$$

If  $x(t)$  is not a constant solution,  $\dot{x}(a) \neq 0$  so that  $x(a)$  must satisfy

$$2 - 6x(a) + 3x(a)^2 = 0 \quad \Leftrightarrow \quad x(a) = 1 \pm \frac{1}{\sqrt{3}}.$$

- (d) Typo in the original version: The material being added into the reactor should be Bo instead of Ct.

Let  $x(t)$  be the number of moles of Bo in the reactor at time  $t$ . The rate of loading is 2 moles per year. Hence  $x(t)$  satisfies  $\dot{x} = -kx + 2$ , where  $k$  is the decay rate of Bo. Since the half life of Bo is 2 years,  $e^{-k \cdot 2} = 1/2$  so that  $k = (\ln 2)/2$ . Therefore we have

$$\dot{x} = -\frac{\ln 2}{2}x + 2.$$

The initial condition is  $x(0) = 0$ .

- (e) The differential equation is linear. Since we have

$$y' + \left(\frac{3}{x}\right)y = x$$

an integrating factor is given by

$$\exp\left(\int \frac{3}{x} dx\right) = \exp(3 \ln x) = x^3.$$

Multiply the above equation by  $x^3$  and integrate:

$$(x^3y)' = x^3y' + 3x^2y = x^4 \quad \Rightarrow \quad x^3y = \frac{1}{5}x^5 + c$$

Since  $y(1) = 1$ , we have  $c = 4/5$  and

$$y = \frac{1}{5}x^2 + \frac{4}{5}x^{-3}.$$

3. (a) Express all complex numbers in polar form:

$$\frac{ie^{2it}}{1+i} = \frac{e^{i\pi/2}e^{2it}}{\sqrt{2}e^{i\pi/4}} = \frac{1}{\sqrt{2}}e^{i(2t+\pi/2-\pi/4)} = \frac{1}{\sqrt{2}}e^{i(2t+\pi/4)}$$

The real part is

$$\operatorname{Re}\left(\frac{ie^{2it}}{1+i}\right) = \frac{1}{\sqrt{2}}\cos\left(2t + \frac{\pi}{4}\right).$$

- (b) The trajectory is an outgoing, clockwise spiral that passes through 1.  
(c) The polar form of  $8i$  is  $8e^{i\pi/2}$ . Its three cubic roots are

$$\begin{aligned} 2e^{i\pi/6} &= 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6} = \sqrt{3} + i, \\ 2e^{i(\pi/6+2\pi/3)} &= 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6} = -\sqrt{3} + i, \\ 2e^{i(\pi/6+4\pi/3)} &= 2e^{3i\pi/2} = -2i. \end{aligned}$$

4. (a) Let  $x_p(t) = at^2 + bt + c$ . Plug it into the left hand side of the equation

$$\begin{aligned} \ddot{x} + 2\dot{x} + 2x &= (2a) + 2(2at + b) + 2(at^2 + bt + c) \\ &= 2at^2 + (4a + 2b)t + (2a + 2b + 2c) \end{aligned}$$

and compare coefficients

$$2a = 1, \quad 4a + 2b = 0, \quad 2a + 2b + 2c = 1.$$

The solution is  $a = 1/2$ ,  $b = -1$ ,  $c = 1$ . Therefore  $x_p(t) = \frac{1}{2}t^2 - t + 1$ .

- (b) The characteristic polynomial is  $p(s) = s^2 + 2s + 2$ . Using the ERF and linearity,

$$x_p(t) = \frac{e^{-2t}}{p(-2)} + \frac{1}{p(0)} = \frac{e^{-2t}}{2} + \frac{1}{2}$$

- (c) Consider the complex equation

$$\ddot{z} + 2\dot{z} + 2z = e^{it}.$$

For any solution  $z_p$ , its imaginary part  $x_p = \text{Im } z_p$  satisfies the real equation

$$\ddot{x} + 2\dot{x} + 2x = \sin t.$$

The ERF provides a particular solution of the complex equation

$$z_p(t) = \frac{e^{it}}{p(i)} = \frac{e^{it}}{1 + 2i} = \frac{e^{it}}{\sqrt{5}e^{i\phi}} = \frac{1}{\sqrt{5}}e^{i(t-\phi)}$$

where  $\phi$  is the polar angle of  $1 + 2i$ . Take the imaginary part of  $z_p$

$$x_p(t) = \text{Im } z_p(t) = \frac{1}{\sqrt{5}} \sin(t - \phi)$$

This is a sinusoidal solution of the real equation. Its amplitude is  $1/\sqrt{5}$ .

- (d) If  $x(t) = t^3$  is a solution, then  $q(t) = \ddot{x} + 2\dot{x} + 2x = 6t + 6t^2 + t^3$ .  
(e) The general solution is  $x(t) = t^3 + x_h(t)$ , where  $x_h(t)$  is a solution of the associated homogeneous equation. Since the characteristic polynomial  $s^2 + 2s + 2$  has roots  $-1 \pm i$ ,

$$x(t) = t^3 + x_h(t) = t^3 + c_1e^{-t} \cos t + c_2e^{-t} \sin t.$$

5. (a) See the formula sheet for the definition of  $\text{sq}(t)$ . The graph of  $f(t)$  is a square wave of period  $2\pi$ . It has a horizontal line segment of height 1 in the range  $-\pi/2 < t < \pi/2$  and a horizontal line segment of height  $-1$  in the range  $\pi/2 < t < 3\pi/2$ .
- (b) Replace  $t$  by  $t + \pi/2$  in the definition of  $\text{sq}(t)$

$$\begin{aligned} f(t) = \text{sq}\left(t + \frac{\pi}{2}\right) &= \frac{4}{\pi} \left[ \sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3} \sin\left(3t + \frac{3\pi}{2}\right) + \frac{1}{5} \sin\left(5t + \frac{5\pi}{2}\right) + \dots \right] \\ &= \frac{4}{\pi} \left( \cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t + \dots \right) \end{aligned}$$

- (c) First consider the complex equation

$$\ddot{z} + z = e^{int} \quad \text{for a positive integer } n.$$

The characteristic polynomial is  $p(s) = s^2 + 1$ . One of the ERFs provides a particular solution of the complex equation

$$\begin{aligned} z_p(t) &= \frac{e^{int}}{p(in)} = \frac{e^{int}}{1 - n^2}, \quad n \neq 1 \\ z_p(t) &= \frac{te^{it}}{p'(i)} = \frac{te^{it}}{2i}, \quad n = 1 \end{aligned}$$

The imaginary parts of these functions

$$\begin{aligned} u_p(t) &= \text{Im} \left( \frac{e^{int}}{1 - n^2} \right) = \frac{\sin nt}{1 - n^2}, \quad n \neq 1 \\ u_p(t) &= \text{Im} \left( \frac{te^{it}}{2i} \right) = -\frac{1}{2} t \cos t, \quad n = 1 \end{aligned}$$

satisfy the imaginary part of the above complex equation, namely

$$\ddot{u} + u = \sin nt.$$

By linearity, a solution of  $\ddot{x} + x = \text{sq}(t)$  is given by

$$x_p(t) = \frac{4}{\pi} \left( -\frac{1}{2} t \cos t + \frac{1}{3} \cdot \frac{\sin 3t}{1 - 3^2} + \frac{1}{5} \cdot \frac{\sin 5t}{1 - 5^2} + \dots \right).$$

6. (a) For  $t < 0$ , the graph is flat on  $t$ -axis.  
 For  $0 < t < 1$ , the graph is flat at 1 unit above  $t$ -axis.  
 For  $1 < t < 3$ , the graph is flat at 1 unit below  $t$ -axis.  
 For  $3 < t < 4$ , the graph is flat at 1 unit above  $t$ -axis.  
 For  $t > 4$ , the graph is flat on  $t$ -axis.
- (b) 
$$\begin{aligned} v(t) &= [u(t) - u(t-1)] - [u(t-1) - u(t-3)] + [u(t-3) - u(t-4)] \\ &= u(t) - 2u(t-1) + 2u(t-3) - u(t-4) \end{aligned}$$

- (c) The graph coincides with  $t$ -axis for all  $t$ , except for two upward spikes at  $t = 0, 3$  and two downward spikes at  $t = 1, 4$ .
- (d)  $\dot{v}(t) = \delta(t) - 2\delta(t - 1) + 2\delta(t - 3) - \delta(t - 4)$
- (e) By the fundamental solution theorem (a.k.a. Green's formula),

$$x(t) = (q * w)(t) = \int_0^t q(t - \tau)w(\tau) d\tau = \int_{a(t)}^{b(t)} w(\tau) d\tau.$$

Now  $q(t - \tau) = 1$  only for  $0 < t - \tau < 1$ , or  $t - 1 < \tau < t$ , and it is zero elsewhere. Therefore the upper limit  $b(t)$  equals  $t$ . The lower limit  $a(t)$  is  $t - 1$  if  $t - 1 > 0$ , or 0 if  $t - 1 < 0$ . In other words,  $a(t) = (t - 1)u(t - 1)$ .

7. (a) The transfer function is  $W(s) = \frac{1}{p(s)} = \frac{1}{2s^2 + 8s + 16}$ .
- (b) The unit impulse response  $w(t)$  is the inverse Laplace transform of  $W(s)$ . In other words,

$$\begin{aligned} \mathcal{L}(w(t)) &= \frac{1}{2s^2 + 8s + 16} = \frac{1}{2[(s + 2)^2 + 4]} \\ \Rightarrow \mathcal{L}(e^{2t}w(t)) &= \frac{1}{2(s^2 + 4)} = \frac{1}{4} \mathcal{L}(\sin 2t) \end{aligned}$$

Therefore  $e^{2t}w(t) = \frac{1}{4} \sin 2t$ , and  $w(t) = \frac{1}{4} e^{-2t} \sin 2t$ .

- (c) Take the Laplace transform of

$$p(D)x = 2\ddot{x}(t) + 8\dot{x}(t) + 16x(t) = \sin t$$

with the initial conditions  $x(0+) = 1$ ,  $\dot{x}(0+) = 2$ . This yields

$$\begin{aligned} 2[s^2X(s) - s - 2] + 8[sX(s) - 1] + 16X(s) &= \frac{1}{s^2 + 1} \\ \Rightarrow X(s) &= \frac{1}{2s^2 + 8s + 16} \left( \frac{1}{s^2 + 1} + 2s + 12 \right) \end{aligned}$$

8. (a) The characteristic polynomial of  $A$  is

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 12 \\ 3 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 - 36 = (\lambda - 8)(\lambda + 4).$$

Therefore the eigenvalues are  $\lambda = 8, -4$ .

- (b) For  $\lambda = 8$ , solve  $(A - 8I)\mathbf{v} = \mathbf{0}$ . Since  $A - 8I = \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix}$ , a solution is  $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .
- For  $\lambda = -4$ , solve  $(A + 4I)\mathbf{v} = \mathbf{0}$ . Since  $A + 4I = \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}$ , a solution is  $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

(c) The following is a fundamental matrix for  $\dot{\mathbf{u}} = B\mathbf{u}$

$$F(t) = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

Then  $e^{tB}$  can be computed as  $F(t)F(0)^{-1}$ .

$$F(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad F(0)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$e^{tB} = F(t)F(0)^{-1} = \begin{bmatrix} e^t & -e^{2t} \\ e^t & e^{2t} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t + e^{2t} & e^t - e^{2t} \\ e^t - e^{2t} & e^t + e^{2t} \end{bmatrix}$$

(d) The general solution of  $\dot{\mathbf{u}} = B\mathbf{u}$  is

$$\mathbf{u}(t) = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ e^{2t} \end{bmatrix} = F(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The given initial condition implies

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = F(0) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = F(0)^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

Therefore the solution of the initial value problem is  $\mathbf{u}(t) = \frac{1}{2} \begin{bmatrix} 3e^t + e^{2t} \\ 3e^t - e^{2t} \end{bmatrix}$ .

9. (a) The phase portrait has the following features:

- All trajectories start at  $(0, 0)$  and run off to infinity.
- There are straight line trajectories along the lines  $y = \pm x$ .
- All other trajectories are tangent to  $y = x$  at  $(0, 0)$ .
- No two trajectories cross each other.

(b)  $\text{Tr } A = a + 1$ ,  $\det A = a + 4$ ,  $\Delta = (\text{Tr } A)^2 - 4(\det A) = (a - 5)(a + 3)$

(i)  $\det A < 0 \Leftrightarrow a < -4$

(ii) not for any  $a$

(iii)  $\Delta > 0$ ,  $\text{Tr } A < 0$  and  $\det A > 0 \Leftrightarrow -4 < a < -3$

(iv)  $\Delta < 0$  and  $\text{Tr } A < 0 \Leftrightarrow -3 < a < -1$ ; counterclockwise

(v)  $\Delta < 0$  and  $\text{Tr } A > 0 \Leftrightarrow -1 < a < 5$

(vi)  $\Delta = 0$  and  $\text{Tr } A > 0 \Leftrightarrow a = 5$

10. (a) The equilibria are the solutions of

$$\dot{x} = x^2 - y^2 = 0, \quad \dot{y} = x^2 + y^2 - 8 = 0.$$

This implies  $(x^2, y^2) = (4, 4)$ , so that  $(x, y) = (2, 2), (2, -2), (-2, 2), (-2, -2)$ .

(b) The Jacobian is  $J(x, y) = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix}$ . In particular,  $J(-2, -2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$ .

- (c) The linearization of the nonlinear system at  $(-2, -2)$  is the linear system  $\dot{\mathbf{u}} = J(-2, -2)\mathbf{u}$ . A computation shows that the eigenvalues of  $J(-2, -2)$  are  $-4 \pm 4i$ . The first component of  $\mathbf{u}(t)$  is of the form

$$c_1 e^{-4t} \cos 4t + c_2 e^{-4t} \sin 4t = A e^{-4t} \cos(4t - \phi).$$

This means  $x(t) \approx -2 + A e^{-4t} \cos(4t - \phi)$  near  $(-2, -2)$ .

- (d) Let  $f(x) = 2x - 3x^2 + x^3$ . The phase line in problem 2(a) shows that  $\dot{x} = f(x)$  has a stable equilibrium at  $x = 1$ .

The linearization of the nonlinear equation at  $x = 1$  is the linear equation  $\dot{u} = f'(1)u = -u$ . Its solutions are  $u(t) = A e^{-t}$ . This means  $x(t) \approx 1 + A e^{-t}$  near  $x = 1$ .

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