18.03 Final Examination

9:00–12:00, May 18, 2010

Your Name	
Recitation Leader	
Recitation Time	

Do not turn the page until you are instructed to do so.

Write your name, your recitation leader's name, and the time of your recitation. Show all your work on this exam booklet. When a particular method is requested you must use it. No calculators or notes may be used, but there is a table of Laplace transforms and other information at the end of this exam booklet. Point values (out of a total of 360) are marked on the left margin. The problems are numbered 1 through 10.

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1. **(a)** and **(b)** concern the tritium that is leaking from the Oyster Creek Nuclear Generating Station into the aquifer in New Jersey at a certain rate, which we will assume is one kilogram per year. The half life of tritium is 12 years.

[6] **(a)** Ignoring other effects (other sources or sinks of the tritium), set up a differential equation for the amount of tritium in the aquifer as a function of time. (For full credit, determine any constants in the equation.)

[6] **(b)** If this leak goes on for a long time, what will the tritium load in the aquifer be? (How many kilograms?)

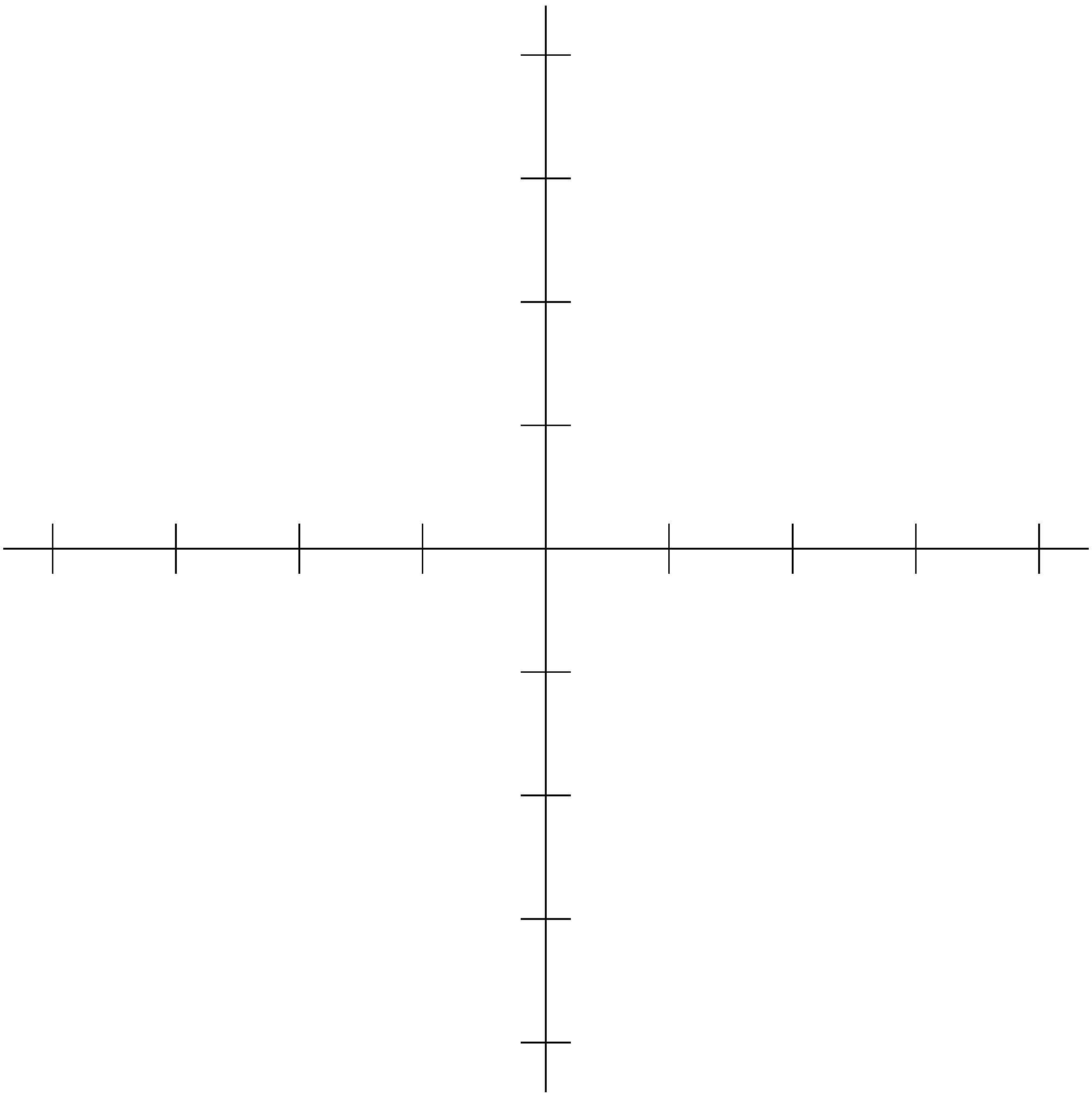
Parts **(c)**–**(g)** of this problem will concern the differential equation $\dfrac{dy}{dx} = x - \dfrac{y^2}{4}$.

[6] **(c)** Let $y(x)$ denote the solution to this equation such that $y(1) = 0$. Use Euler's method with step size $\frac{1}{2}$ to estimate $y(2)$.

1. Continuing with $\frac{dy}{dx} = x - \frac{y^2}{4}$

[3] **(d)** Sketch the isoclines for slopes $m = -1$, $m = 0$, and $m = 1$, on the plane below.

[3] **(e)** On the same plane, sketch the graph of the solution $y(x)$ with $y(1) = 0$.



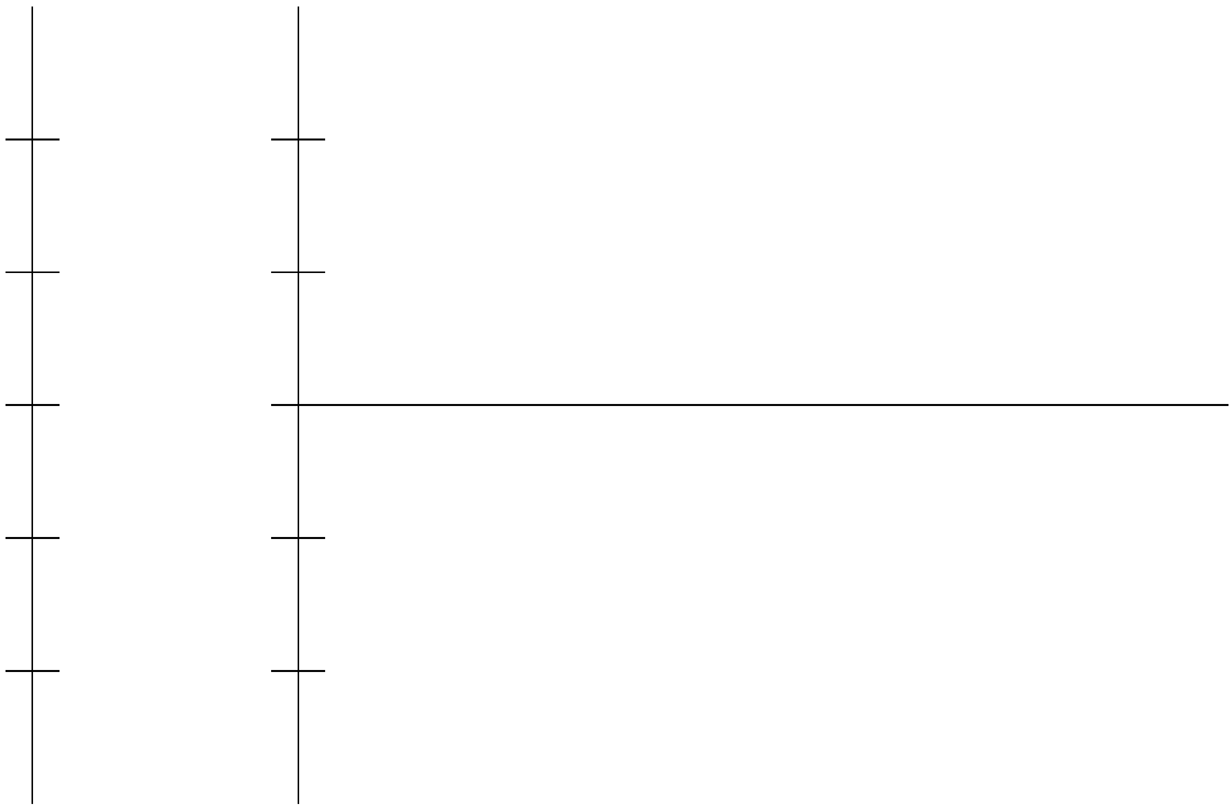
[6] **(f)** For this same solution, suppose that $y(x)$ achieves a minimum at $x = a$. What is $y(a)$ (in terms of a)?

[6] **(g)** Estimate the value of $y(100)$.

2. In **(a)**–**(c)** we consider the autonomous equation $\dot{x} = x^3 - x^2 - 2x$.

[10] **(a)** On the vertical line below, sketch the phase line of this equation.

[10] **(b)** Sketch the graphs of some solutions. Be sure to include at least one solution with values in each interval above, below, and between the critical points.



[6] **(c)** Suppose $x(0)$ is quite small, say 0.1. For $t > 0$, $x(t)$ is best approximated by $(0.1)e^{at}$ for what value of a?

[10] **(d)** Solve the initial value problem $x\dfrac{dy}{dx} + 2y = -\dfrac{\sin(x)}{x}$, $y(\pi) = 0$.

[6] **3. (a)** Find a complex number r (expressed as $r = a + bi$ with a, b real) and a positive real number ω such that $\operatorname{Re}\left(\frac{e^{i\omega t}}{r}\right) = 2\cos\left(2t - \frac{\pi}{4}\right)$.

[6] **(b)** Express the cube roots of $-8i$ in the form $a + bi$ (with a and b real).

[8] **(c)** What is the pseudoperiod of a nonzero solution to $\ddot{x} + 2\dot{x} + 10x = 0$?

[8] **3.** (continued) **(d)** At what circular frequency $\omega = \omega_r$ does the sinusoidal solution to $\ddot{x} + 2\dot{x} + 10x = \cos(\omega t)$ have the largest amplitude?

[8] **(e)** At what circular frequency $\omega = \omega_p$ is the phase lag of the solution to $\ddot{x}+2\dot{x}+10x = \cos(\omega t)$ equal to $\frac{\pi}{2}$?

4. Let $p(D) = D^2 + 4D + 8I$.

[10] **(a)** Find one solution to $p(D)x = 8t^2$.

[10] **(b)** Find one solution to $p(D)x = e^{-t}$.

In **(c)** and **(d)**, suppose that $x = t^3$ is a solution to $p(D)x = q(t)$ (where still $p(D) = D^2 + 4D + 8I$).

[8] **(c)** What is $q(t)$?

[8] **(d)** What is the solution to $p(D)x = q(t)$ such that $x(0) = 0$ and $\dot{x}(0) = 2$?

[15] **5.** **(a)** Find a periodic solution to $\ddot{x} + \omega_n^2 x = \mathrm{sq}(2t)$ if one exists.

(b)–(c) concern the function $f(t)$, periodic of period 2π, with $f(t) = |t| - \frac{\pi}{2}$ for $|t| \leq \pi$.

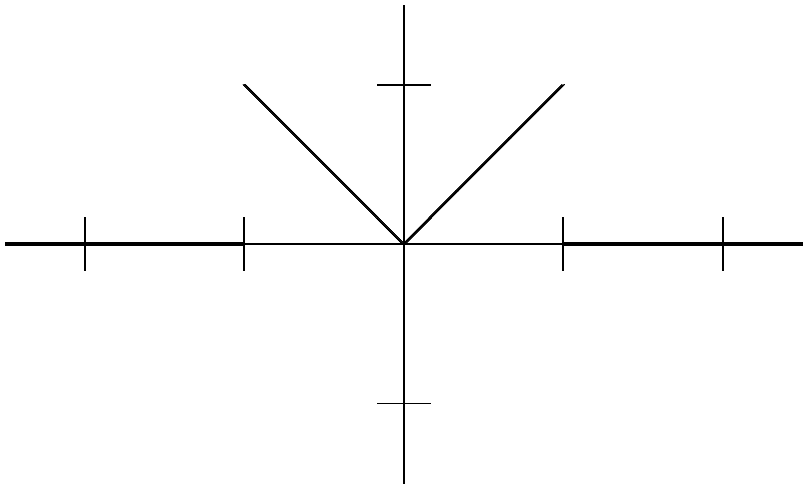
[6] **(b)** Is $f(t)$ even, odd, or neither?

What is its average value?

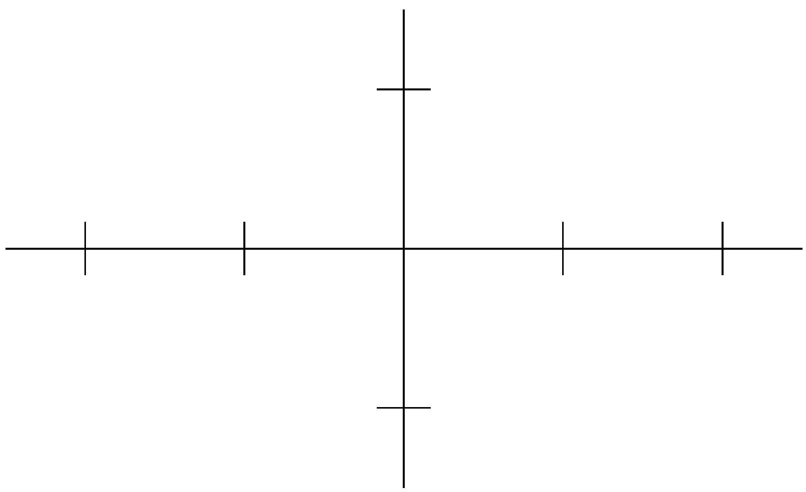
Graph $f(t)$.

[15] **(c)** What is the Fourier series of $f(t)$?

6. (a)–(b) The function $f(t)$ is defined by the following graph, in which the hashmarks are at unit spacing.



[6] **(a)** Graph the generalized derivative $f'(t)$.



[8] **(b)** Write a formula for $f'(t)$, using step and delta functions as necessary. Identify in your formula the regular part $f'_r(t)$ and the singular part $f'_s(t)$.

6. (continued) Suppose that the unit impulse response of a certain operator $p(D)$ is $w(t) = u(t)e^{-t}\sin(2t)$.

[8] **(c)** Please calculate the solution to $p(D)x = u(t)e^{-t}$, with rest initial conditions.

[6] **(d)** What is the solution to $p(D)x = \delta(t-1)$ with rest initial conditions?

[8] **(e)** What is the characteristic polynomial $p(s)$?

7. **(a)–(c)** concern the operator $p(D) = D^3 + D$.

[8] **(a)** What is the transfer function of the operator $p(D)$?

[10] **(b)** What is the unit impulse response of this operator?

[10] **(c)** What is the Laplace transform $X(s)$ of the solution to $p(D)x = \cos(2t)$ with rest initial conditions?

[8] **(d)** Sketch the pole diagram of $\dfrac{(s+1)e^{-s}}{s^3 + 2s^2 + 2s}$

8. In **(a)** and **(b)**, $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$.

[8] **(a)** What are the eigenvalues of A?

[8] **(b)** For each eigenvalue, find a nonzero eigenvector.

(c)–(e) concern a certain 2×2 matrix B, whose eigenvalues are known to be 2 and -2.

[6] **(c)** What is the trace of B?

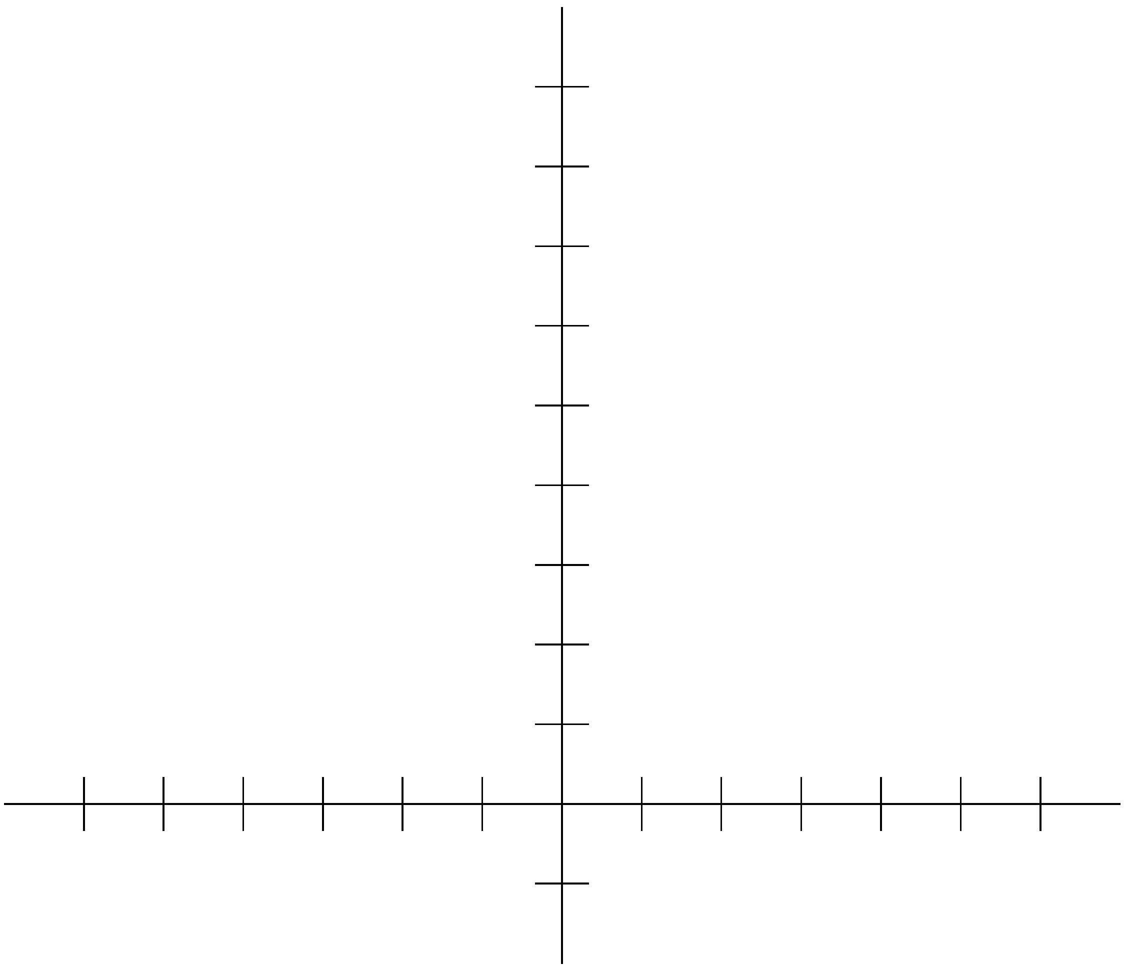
What is the determinant of B?

[8] **(d)** Suppose also that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for value 2 and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector for value -2. Calculate e^{Bt}.

[6] **(e)** What is the solution to $\dot{\mathbf{u}} = B\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

9. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & a \end{bmatrix}$, and consider the homogeneous linear system $\dot{\mathbf{u}} = A\mathbf{u}$.

[6] **(a)** On the (Tr,Det) plane below, sketch the critical parabola (where the matrix has repeated eigenvalues) and sketch line corresponding to the matrices A as a varies.



[30] **(b)** For each of the following conditions, indicate the values of a (if any) for which the phase portrait satisfies the condition.

(i) Saddle

(ii) Stable node = nodal sink

(iii) Unstable node = nodal source

(iv) Stable spiral = spiral sink

(v) Unstable spiral = spiral source

(vi) Stable defective node = defective nodal sink

(vii) Unstable defective node = defective nodal source

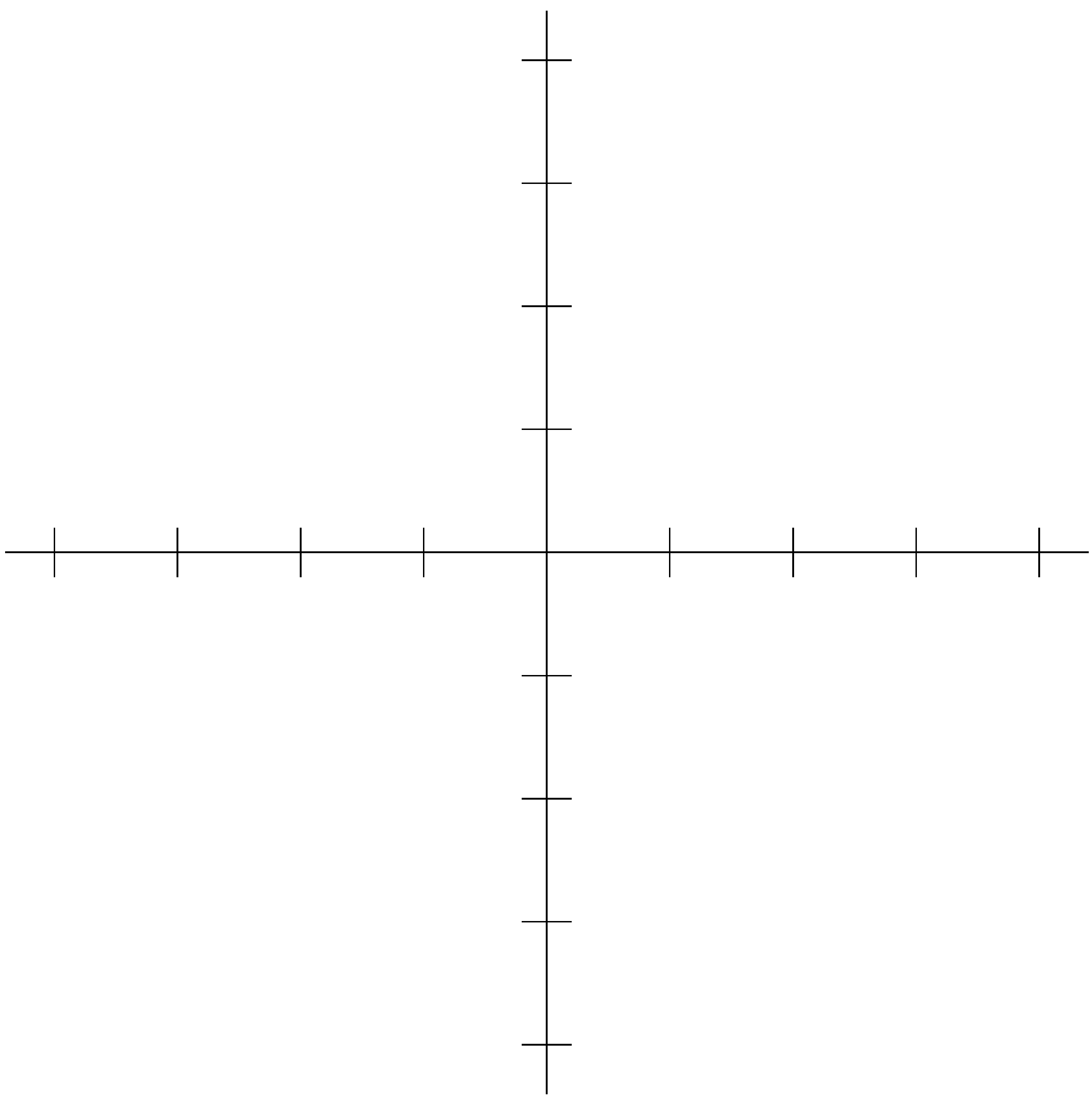
(viii) Star

(ix) Center

(x) Degenerate case

10. This problem concerns the nonlinear autonomous system $\begin{cases} \dot{x} = (y+1)(y-x+1) \\ \dot{y} = (x-3)(x+y-1) \end{cases}$.

[12] **(a)** Find the equilibria of this system and plot them on the (x, y) plane below.



10. (continued) $\begin{cases} \dot{x} = (y+1)(y-x+1) \\ \dot{y} = (x-3)(x+y-1) \end{cases}$.

[12] **(b)** One equilibrium is at $(1, 0)$. Find the Jacobian matrix at this equilibrium.

[12] **(c)** The equilibrium at $(1, 0)$ is a stable spiral. For large t, the solutions which converge to this equilibrium have y-coordinate well-approximated by the function $Ae^{at}\cos(\omega t - \phi)$ for some constants A, ϕ, a, and ω. Some of these constants depend upon the particular solution, and some are common to all solutions of this type. Find the values of the ones which are common to all such solutions.

Operator Formulas

• Exponential Response Formula: $x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$ provided $p(r) \neq 0$.

• Resonant Response Formula: If $p(r) = 0$ then $x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$ provided $p'(r) \neq 0$.

Defective matrix formula

If A is a defective 2×2 matrix with eigenvalue λ_1 and nonzero eigenvector $\mathbf{v}_1$, then you can solve for $\mathbf{w}$ in $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1$ and $\mathbf{u} = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{w})$ is a solution to $\dot{\mathbf{u}} = A\mathbf{u}$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}(s) >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$ [generalized derivative]
$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ [$f(t)$ continuous for $t > 0$]

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = \frac{1}{p(s)}$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}[\delta_a(t)] = e^{-as}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$$

Fourier series $f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$