18.03 Hour Exam III

April 23, 2010

Your Name
Your Recitation Leader's Name
Your Recitation Time

Problem	Points
1	
2	
3	
4	
5	
Total	

Do not open this booklet till told to do so. There are five problems. Use your test-taking skills—be sure you get to all the problems. Do all your work on these pages. No calculators or notes may be used. The point value (out of 100) of each problem is marked in the margin. Solutions will be available on the web after 4:00 today, and at recitation.

There is a page of formulas at the back of the exam.

1. A certain periodic function has Fourier series

$$f(t) = 1 + \frac{\cos(\pi t)}{2} + \frac{\cos(2\pi t)}{4} + \frac{\cos(3\pi t)}{8} + \frac{\cos(4\pi t)}{16} + \cdots$$

[4] **(a)** What is the minimal period of $f(t)$?

[4] **(b)** Is $f(t)$ even, odd, neither, or both?

[8] **(c)** Please give the Fourier series of a periodic solution (if one exists) of

$$\ddot{x} + \omega_n^2 x = f(t)$$

[4] **(d)** For what values of ω_n is there no periodic solution?

2. Let $f(t) = (u(t+1) - u(t-1))t$.

[6] **(a)** Sketch a graph of $f(t)$.

[6] **(b)** Sketch a graph of the generalized derivative $f'(t)$.

[8] **(c)** Write a formula for the generalized derivative $f'(t)$, and identify in your formula the regular part $f'_r(t)$ and the singular part $f'_s(t)$.

3. Let $p(D)$ be the operator whose unit impulse response is given by $w(t) = e^{-t} - e^{-3t}$.

[10] **(a)** Using convolution, find the unit step response of this operator: the solution to $p(D)v = u(t)$ with rest initial conditions.

[5] **(b)** What is the transfer function $W(s)$ of the operator $p(D)$?

[5] **(c)** What is the characteristic polynomial $p(s)$?

[10] **4 (a)** Find a generalized function $f(t)$ with Laplace transform $F(s) = \dfrac{e^{-s}(s-1)}{s}$.

[10] **(b)** Find a function $f(t)$ with Laplace transform $F(s) = \dfrac{s+10}{s^3+2s^2+10s}$.

5. Let $W(s) = \dfrac{s+10}{s^3+2s^2+10s}$.

[10] **(a)** Sketch the pole diagram of $W(s)$.

[10] **(b)** If $p(D)$ is the operator with transfer function $W(s)$, what is the Laplace transform of the solution, with rest initial conditions, of $p(D)x = \sin(2t)$?

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\operatorname{Re} s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$, where $f'(t)$ denotes the generalized derivative.

$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ if $f(t)$ is continuous for $t > 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\operatorname{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\operatorname{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$