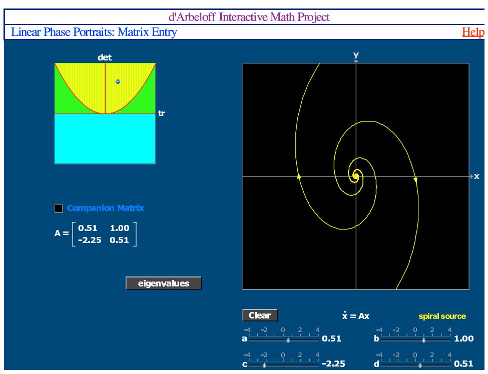
18.03 Problem Set 9: Solutions

Part I: 34. 4; **35.** 10; **36.** 6.

(a) [18] $A = \begin{bmatrix} 0.5 & 1 \\ -2.25 & 0.5 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - \lambda + 2.5$, and eigenvalues $\frac{1 \pm 3i}{2}$. An eigenvector for $\lambda_1 = \frac{1+3i}{2}$ satisfies $(A - \lambda_1 I)\mathbf{v_1} = \mathbf{0}$, that is, $\begin{bmatrix} -3i/2 & 1 \\ -9/4 & -3i/2 \end{bmatrix} \mathbf{v_1} = \mathbf{0}$. One choice is $\mathbf{v_1} = \begin{bmatrix} 1 \\ 3i/2 \end{bmatrix}$. The normal mode is then $e^{(1+3i)t/2} \begin{bmatrix} 1 \\ 3i/2 \end{bmatrix}$, which has real and imaginary parts $\mathbf{u_1} = e^{t/2} \begin{bmatrix} \cos(3t/2) \\ -(3/2)\sin(3t/2) \end{bmatrix}$ and $\mathbf{u_2} = e^{t/2} \begin{bmatrix} \sin(3t/2) \\ (3/2)\cos(3t/2) \end{bmatrix}$.

The initial condition is $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which, conveniently, is satisfied by $\mathbf{u_1}$. Since $\dot{\mathbf{u}} = A\mathbf{u}$, we find $\dot{\mathbf{u}}(0) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -2.25 \end{bmatrix}$. So $\dot{x}(0) = 0.5$, and the weasel population is increasing at $t = 0$ while the number of voles is decreasing. $y(t) = 0$ occurs next when $3t/2 = \pi$, or $t = 2\pi/3$.



The graphs of $x(t) = e^{t/2}\cos(3t/2)$ and $y(t) = -(3/2)e^{t/2}\sin(3t/2)$ are "anti-damped" sinusoids, with increasing amplitude. The relevant trajectory is the one crossing the positive x axis half way out. The values of $\mathbf{u}(t)$ are $\mathbf{u}\left(-\frac{2\pi}{3}\right) = e^{-\pi/3} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{u}\left(-\frac{\pi}{3}\right) = e^{-\pi/6} \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$, $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}\left(\frac{\pi}{3}\right) = e^{\pi/6} \begin{bmatrix} 0 \\ -3/2 \end{bmatrix}$, $\mathbf{u}\left(\frac{2\pi}{3}\right) = e^{\pi/3} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

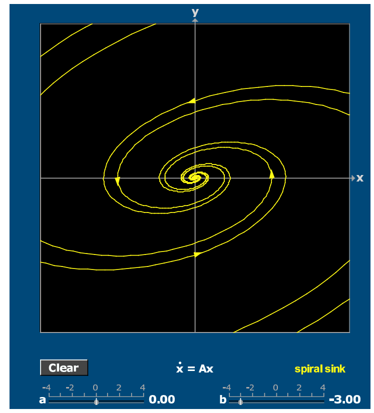
(b) [8] With $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, $p_A(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so we have a repeated eigenvalue $\lambda_1 = 1$. To find an eigenvector form $A - \lambda_1 I = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. A nonzero eigenvector is given (for any b) by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If $b \neq 0$, the eigenvectors for value λ_1 are exactly the multiples of $\mathbf{v}$ (the matrix is defective), but for $b = 0$, $A = I$ and any vector is an eigenvector (the matrix is complete). When $b \neq 0$, the normal modes are $e^t \begin{bmatrix} c \\ 0 \end{bmatrix}$, for c a real constant. When $b = 0$, the normal modes are $e^t\mathbf{v}$ for any vector $\mathbf{v}$. When $b \neq 0$, we must solve $(A - \lambda_1 I)\mathbf{w} = \mathbf{v_1}$, that is, $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The solution is $\mathbf{w} = \begin{bmatrix} 0 \\ 1/b \end{bmatrix}$, so the extra solution is $\mathbf{u_2} = e^{\lambda_1 t}(t\mathbf{v_1} + \mathbf{w}) = e^t \begin{bmatrix} t \\ 1/b \end{bmatrix}$.

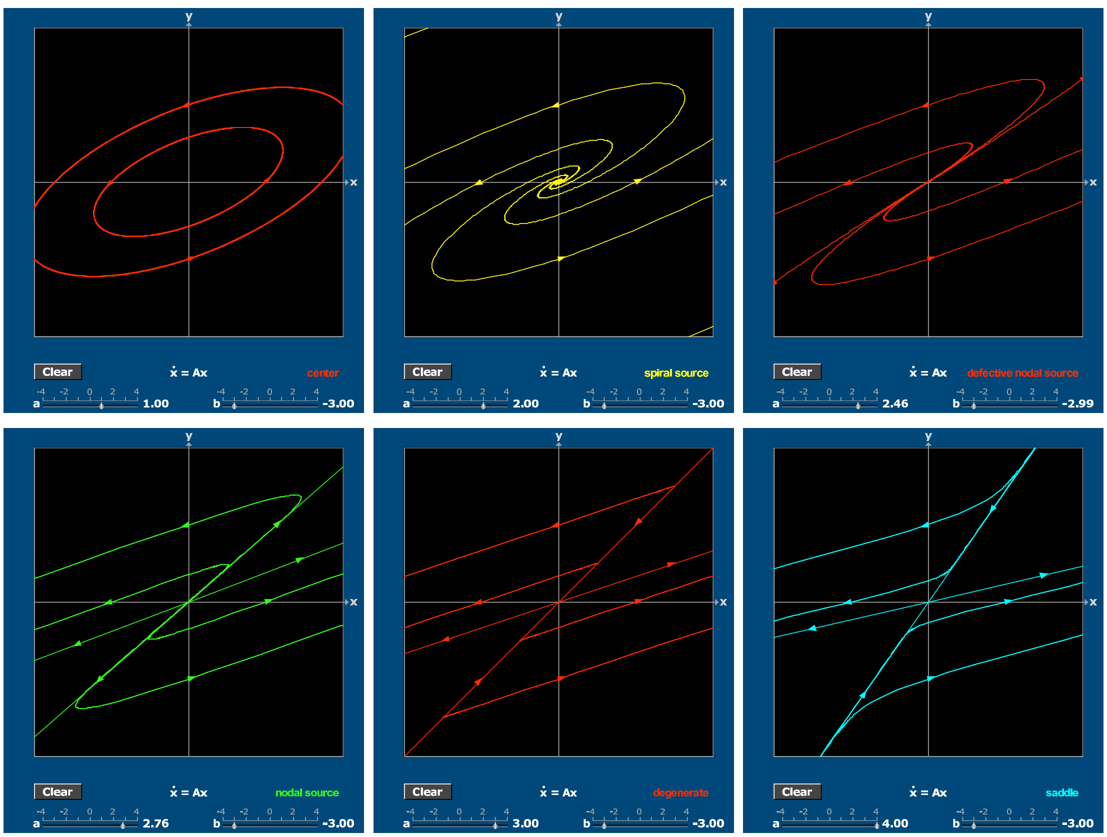
35. (a) [4] $\text{tr}A = a - 1$, $\det A = 3 - a$, so $\text{tr}A = 2 - \det A$. $\det A = 0$ when $a = 3$. $\text{tr}A = 0$ when $a = 1$. $\det A = (\text{tr}A)^2/4$ when $a^2 + 2a - 11 = 0$ or $a = -1 \pm 2\sqrt{3}$, i.e. $a \simeq -4.4641$ and $a = 2.4641$.

(c) [4] Diagram showing: $a < -1 - 2\sqrt{3}$—stable node = nodal sink
$a = -1 - 2\sqrt{3}$—defective stable node = defective nodal sink
$-1 - 2\sqrt{3} < a < 1$—counterclockwise stable spiral = spiral sink

$a = 1$—counterclockwise center
$1 < a < -1 + 2\sqrt{3}$–counterclockwise unstable spiral = spiral source
$a = 1 + 2\sqrt{3}$—unstable defective node = defective nodal source
$1 + 2\sqrt{3} < a < 3$—unstable node = nodal source
$a = 3$—unstable degenerate comb
$3 < a$—saddle

(b)-(c) [18] Here are pictures for $a = 0, 1, 2, -1 + 2\sqrt{3}, 2.75, 3, 4$. ($a = -2\sqrt{3}$ omitted.) The picture for some $a < -1 - 2\sqrt{3}$ would show a nodal sink, and that for $a = -1 - 2\sqrt{3}$ would show a defective nodal sink.





36. (a) [9] With $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $p_A(\lambda) = \lambda^2 - 2a\lambda + (a^2+b^2) = (\lambda - a)^2 + b^2$, so the eigenvalues are $a \pm bi$. An eigenvector for $\lambda_1 = a + bi$ is given by $\mathbf{v_1}$ such that $\begin{bmatrix} -bi & -b \\ b & -bi \end{bmatrix} \mathbf{v_1} = \mathbf{0}$, and we can take $\mathbf{v_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. The corresponding normal mode is $e^{(a+bi)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Its real and imaginary parts give linearly independent real solutions, $e^{at} \begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix}$ and $e^{at} \begin{bmatrix} \sin(bt) \\ \cos(bt) \end{bmatrix}$.

So a fundamental matrix is given by $\Phi(t) = e^{at}\begin{bmatrix} \cos(bt) & \sin(bt) \\ \sin(bt) & -\cos(bt) \end{bmatrix}$. $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so $e^{At} = \Phi(t)\Phi(0)^{-1} = e^{at}\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$.

$A(e^{(a+bi)t}) = A(e^{at}(\cos(bt) + i\sin(bt))) = e^{at}\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} = e^{A(a+bi)t}$.

(b) [9] $s^2 + 2s + 2 = (s+1)^2 + 1$ so the roots of the characteristic polynomial are $-1 \pm i$. Basic solutions are given by $y_1 = e^{-t}\cos(t)$ and $y_2 = e^{-t}\sin(t)$. (I write y instead of x because the problem wrote x for the normalized solutions.) $y_1(0) = 1$, $\dot{y}_1(0) = -1$, $y_2(0) = 0$, $\dot{y}_2(0) = 1$. So $x_1 = y_1 + y_2$ and $x_2 = y_2$ form a normalized pair of solutions: $x_1(t) = e^{-t}(\cos t + \sin t)$, $x_2(t) = e^{-t}\sin t$.

The companion matrix is $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$. Its characteristic polynomial is the same, $\lambda^2+2\lambda+2$, so its eigenvalues are the same, $-1 \pm i$. An eigenvector for value $-1+i$ is given by $\mathbf{v_1}$ such that $\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\mathbf{v_1} = \mathbf{0}$. We can take $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$. The corresponding normal mode is $e^{(-1+i)t}\begin{bmatrix} 1 \\ -1+i \end{bmatrix}$, which has real and imaginary parts $\mathbf{u_1} = e^{-t}\begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix}$ and $\mathbf{u_2} = e^{-t}\begin{bmatrix} \sin t \\ -\sin t + \cos t \end{bmatrix}$. $\Phi(t) = [\mathbf{u_1} \quad \mathbf{u_2}]$ has $\Phi(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, so $e^{At} = \Phi(t)\Phi(0)^{-1} = e^{-t}\begin{bmatrix} \cos t + \sin t & \sin t \\ -2\sin t & -\sin t + \cos t \end{bmatrix}$. The top entries coincide with x_1 and x_2 computed above.

(c) [9] (i) $\mathbf{u_1} = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u_1}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$. Thus $c_1 = 2$ and $c_2 = -1$: $\mathbf{u_1} = \begin{bmatrix} 2e^{3t} - e^{2t} \\ 2e^{3t} - 2e^{2t} \end{bmatrix}$. Start again for $\mathbf{u_2}$: $\mathbf{u_2} = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{u_2}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{bmatrix}$. Thus $c_1 = -1$ and $c_2 = 1$: $\mathbf{u_2} = \begin{bmatrix} -e^{3t} + e^{2t} \\ -e^{3t} + 2e^{2t} \end{bmatrix}$.

(ii) We have just computed the columns of the exponential matrix:
$e^{At} = \begin{bmatrix} 2e^{3t} - e^{2t} & -e^{3t} + e^{2t} \\ 2e^{3t} - 2e^{2t} & -e^{3t} + 2e^{2t} \end{bmatrix}$.

(iii) The matrix A has eigenvalues 3 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The top entries give the equations $a + b = 3$ and $a + 2b = 2$, which imply $a = 4$, $b = -1$. The bottom entries give the equations $c + d = 3$, $c + 2d = 4$, which imply $c = 2$, $d = 1$. Thus $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.