18.03 Problem Set 8: Part II Solutions

Part I points: 29. 4, **30.** 0, **32.** 8, **33.** 4.

Part II solutions:

29. (a) [8] (1) $\{1, i, -i\}$: For large t, these functions have exponential growth rate e^t. (This means that for any $a < 1 < c$, $e^{at} < |f(t)| < e^{ct}$ for large t.) They also show a small oscillation of approximately constant amplitude and circular frequency 1. Examples: $f(t) = ae^t + b\sin(t)$ $(a, b \neq 0)$ with $F(s) = \dfrac{a}{s-1} + \dfrac{b}{s^2+1}$.

(2) $\{-1+4i, -1-4i\}$: For large t, these functions show exponential decay like e^{-t}, and oscillate with circular frequency 4. Examples: $f(t) = ae^{-t}\sin(4t)$ $(a \neq 0)$ with $F(s) = \dfrac{4a}{(s+1)^2+16}$.

(3) $\{-1\}$: For large t, these functions decay like e^{-t} and do not oscillate. Examples: $f(t) = ae^{-t}$ $(a \neq 0)$ with $F(s) = \dfrac{a}{s+1}$.

(4) No poles: For large t, these functions decay to zero faster than any exponential. Examples: $f(t) = a\delta(t-b)$ $(a \neq 0, b \geq 0)$ with $F(s) = ae^{-bt}$, or $f(t) = a(u(t) - u(t-b))$ $(a \neq 0, b > 0)$ with $F(s) = a\dfrac{1-e^{-bs}}{s}$.

(b) (i) [4] Method I: For $t > 0$, $w(t)$ is a solution to the homogeneous equation. The roots must be $-\frac{1}{2} \pm \frac{3}{2}i$, so $p(s) = m\left(s - (-\frac{1}{2} + \frac{3}{2}i)\right)\left(s - (-\frac{1}{2} - \frac{3}{2}i)\right) = m(s^2+s+\frac{5}{2})$. To find m, remember that $\dot{w}(0+) = \frac{1}{m}$ (for a second order system). $\dot{w}(t) = u(t)e^{-t/2}(\frac{3}{2}\cos(3t/2) - (-\frac{1}{2}\sin(3t/2))$, so $\dot{w}(0+) = \frac{3}{2}$, $m = \frac{2}{3}$, and $p(s) = \frac{2}{3}(s^2+s+\frac{5}{2})$.

Method II: $W(s) = \mathcal{L}[w(t)] = \dfrac{3/2}{(s+(1/2))^2+(9/4)} = \dfrac{1}{(2/3)(s^2+s+(5/2))}$,
so $p(s) = \frac{2}{3}(s^2+s+\frac{5}{2})$.

(ii) [4] $\{-\frac{1}{2} \pm \frac{3}{2}i\}$.

(iii) [4] This is a throwback problem. The complex gain is $W(i\omega) = \dfrac{3/2}{((5/2)-\omega^2)+i\omega}$, so the gain is $|W(i\omega)| = \dfrac{3/2}{\sqrt{((5/2)-\omega^2)^2+\omega^2}}$. This is maximized when the denominator, or its square $(\frac{5}{2}-\omega^2)^2+\omega^2$, is minimized. The derivative with respect to ω is
$2(\frac{5}{2}-\omega^2)(-2\omega)+2\omega$, which has roots at $\omega = 0$ and $\omega = \pm\sqrt{2}$. So $\omega_r = \sqrt{2}$ is the worst frequency.

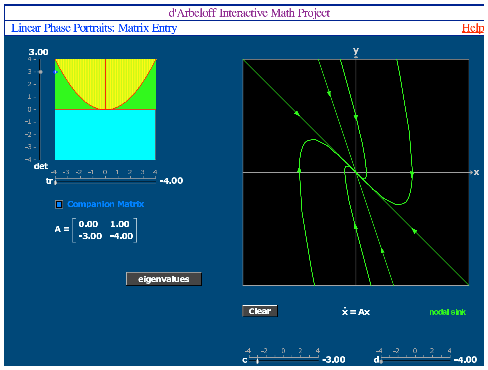
(iv) [6] The yellow box lies in the plane above the imaginary axis. The base is the amplitude response curve. The green box lies in the plane above the real axis. Its top lies on the graph of $|W(s)|$, and its base is the real axis. The red arrows lie above the poles of $W(s)$. The yellow diamonds are located at $(\pm i\omega, |W(i\omega)|)$, and represent the chosen value of the input circular frequency and the corresponding gain.

32. (a) [8] $p(s) = s^2+4s+3 = (s+2)^2-1$ has roots $r_1 = -1$ and $r_2 = -3$. Basic solutions are given by $x_1 = e^{-t}$ and $x_2 = e^{-3t}$. (The order is not determined, and in fact any other pair of linearly independent solutions count as "basic.") $\dot{x}_1 = -e^{-t}$, $\dot{x}_2 = -3e^{-3t}$.

$p(s) = s^2 + s + \frac{5}{2} = (s + \frac{1}{2})^2 + \frac{9}{4}$ has roots $r = -\frac{1}{2} \pm \frac{3}{2}i$. Basic solutions are given by $x_1 = e^{-t/2}\cos(\frac{3}{2}t)$ and $x_2 = e^{-t/2}\sin(\frac{3}{2}t)$. (Same caveats as above.) $\dot{x}_1 = e^{-t/2}(-\frac{1}{2}\cos(\frac{3}{2}t) - \frac{3}{2}\sin(\frac{3}{2}t))$. $\dot{x}_2 = e^{-t/2}(-\frac{1}{2}\sin(\frac{3}{2}t) + \frac{3}{2}\cos(\frac{3}{2}t))$.

(b) [4] $\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -\frac{5}{2} & -1 \end{bmatrix}$.

(c) [4]

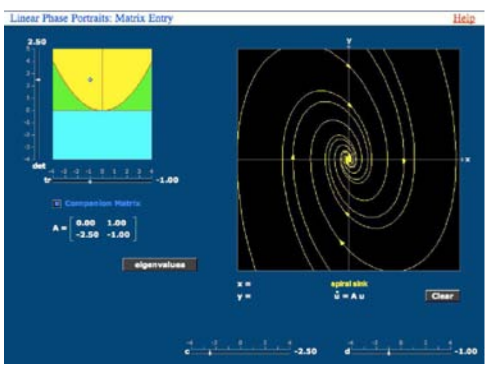


The ray containing $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponds to x_1; the ray containing $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ corresponds to x_2.

(d) [4] The solution passing through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t = 0$ is $\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ where x is the solution to $\ddot{x} + 4\dot{x} + 3x = 0$ with $x(0) = 1$, $\dot{x}(0) = 0$. The general solution $x(t) = c_1 e^{-t} + c_2 e^{-3t}$ has $x(0) = c_1 + c_2$ and $\dot{x}(0) = -c_1 - 3c_2$, so $c_1 + c_2 = 1$ and $-c_1 - 3c_2 = 0$. Thus $c_2 = -\frac{1}{2}$ and $c_1 = \frac{3}{2}$ and $x(t) = \frac{1}{2}(3e^{-t} - e^{-3t})$, so $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3e^{-t} - e^{-3t} \\ -3e^{-t} + 3e^{-3t} \end{bmatrix}$. Description of the graph of $x(t)$: For $t << 0$ it is very negative and increasing. $x(t) = 0$ for $t = -\frac{\ln 3}{2}$. It reaches a maximum $x(0) = 1$, and then falls back through an inflection point to become asymptotic to $x = 0$ as $t \to \infty$.

(e) [4] $\mathbf{u}(t) = \frac{1}{2}\begin{bmatrix} 3e^{-(t-a)} - e^{-3(t-a)} \\ -3e^{-(t-a)} + 3e^{-3(t-a)} \end{bmatrix}$.

(f) [8]



The trajectory of interest is the spiral passing through $(0, 1)$.

The solution $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ passing through $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ at $t = 0$ is given by the solution $x(t)$ of $\ddot{x} + \dot{x} + \frac{5}{2}x = 0$ with $x(0) = 0$, $\dot{x}(0) = 1$. The general solution $x(t) = e^{-t/2}(c_1\cos(\frac{3}{2}t) + c_2\sin(\frac{3}{2}t))$ has $x(0) = c_1$ and $\dot{x}(0) = -\frac{1}{2}c_1 + \frac{3}{2}c_2$, so $c_1 = 0$ and $c_2 = \frac{2}{3}$. Thus $x(t) = \frac{2}{3}e^{-t/2}\sin(\frac{3}{2}t)$, $\dot{x}(t) = e^{-t/2}(\cos(\frac{3}{2}t) - \frac{1}{3}\sin(\frac{3}{2}t))$, and $\mathbf{u}(t) = e^{-t/2}\begin{bmatrix} \frac{2}{3}\sin(\frac{3}{2}t) \\ \cos(\frac{3}{2}t) - \frac{1}{3}\sin(\frac{3}{2}t) \end{bmatrix}$. This passes through the y axis when $x(t) = 0$, i.e. when $\sin(\frac{3}{2}t) = 0$, which is when t is an integral multiple of $2\pi/3$.

Both $x(t)$ and $y(t)$ are damped sinuosoids, and $x(0) = 0$ and $\dot{x}(0) = 1$, $y(0) = 1$, $\dot{y}(0) = -1$.

(g) [4] The velocity vector of $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ is $\dot{\mathbf{u}}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}$. When $\dot{x}(t) = 0$, then, $\dot{\mathbf{u}}(t) = \begin{bmatrix} 0 \\ \ddot{x}(t) \end{bmatrix}$, which is vertical. Alternatively, $\dot{\mathbf{u}} = A\mathbf{u}$ and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$, so if $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ then $\dot{\mathbf{u}} = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 0 \\ c \end{bmatrix}$, which is vertical.

33. (a) [8] $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{bmatrix} = -\lambda(-4-\lambda)+3 = \lambda^2+4\lambda+3$ (the same as the characteristic polynomial of $D^2+4D+3I$!), which has roots $\lambda_1 = -1$, $\lambda_2 = -3$. A vector $\mathbf{v}$ is an eigenvector for eigenvalue λ when $(A - \lambda I)\mathbf{v} = \mathbf{0}$. With $\lambda = -1$ this gives $\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \mathbf{v_1} = \mathbf{0}$, so a nonzero eigenvalue for $\lambda = -1$ is $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero multiple. With $\lambda = -3$ this gives $\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \mathbf{v_1} = \mathbf{0}$, so a nonzero eigenvalue for $\lambda = -3$ is $\mathbf{v_2} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ or any nonzero multiple.

The eigenlines are the straight lines sketched on the plane in **32(c)** above. For the eigenline for eigenvalue -1 the basic solution is $\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$, so the general solution along that eigenline is $c \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ for a constant c. For the eigenline for eigenvalue -3 the basic solution is $\begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$, so the general solution along that eigenline is $c \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$ for a constant c.

These are the solutions found in **3(c)**.

(b) [8] $p_A(\lambda) = (2-\lambda)(-3-\lambda) + 4 = \lambda^2 + \lambda - 2$, so the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1$. (The order doesn't matter.) For $\lambda = -2$ a nonzero eigenvector is given by $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and for $\lambda = 1$ by $\mathbf{v_2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. The general solution along the first eigenline is $ce^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and along the second is $ce^{t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

(c) [4]