18.03 Problem Set 7: Part II Solutions

Part I points: 26. 6, **27.** 10, **28.** 12.

I.26. $e^{-t}\sin(3t) = \frac{1}{2i}\left(e^{(-1+3i)t} - e^{(-1-3i)t}\right)$, so $\mathcal{L}[e^{-t}\sin(3t)] =$
$\frac{1}{2i}\left(\frac{1}{s-(-1+3i)} - \frac{1}{s-(-1-3i)}\right) = \frac{1}{2i}\frac{(s+1+3i)-(s+1-3i)}{(s+1)^2+9} = \frac{3}{(s+1)^2+9}$.

26. (a) [10] $G(s) = \int_0^\infty f(at)e^{-st}\,dt$. To make this look more like $F(s) = \int_0^\infty f(t)e^{-st}\,dt$, make the substitution $u = at$. Then $du = a\,dt$ and

$$G(s) = \int_0^\infty f(u)e^{-su/a}\,\frac{du}{a} = \frac{1}{a}\int_0^\infty f(u)e^{-(s/a)u}\,du = \frac{1}{a}F\left(\frac{s}{a}\right).$$

For example, take $f(t) = t^n$, so $F(s) = \frac{n!}{s^{n+1}}$, $g(t) = (at)^n = a^n t^n$, $G(s) = \frac{a^n n!}{s^{n+1}}$. Now compute $\frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}\frac{n!}{(s/a)^{n+1}} = \frac{a^{n+1}}{a}\frac{n!}{s^{n+1}} = \frac{a^n n!}{s^{n+1}} = G(s)$.

(b) [10] Compute $F(s)G(s) = \int_0^\infty\int_0^\infty f(x)e^{-sx}g(y)e^{-sy}\,dxdy = \iint_R f(x)g(y)e^{-s(x+y)}\,dxdy$, where R is the first quadrant. The suggested substitution is $x = t-\tau$, $y = \tau$. To convert to these coordinates, note that the Jacobian is $\det\begin{bmatrix}\frac{\partial x}{\partial t} & \frac{\partial x}{\partial \tau}\\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \tau}\end{bmatrix} = \det\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = 1$ For fixed t, τ ranges over numbers between 0 and t, and t ranges over positive numbers. Since $x+y=t$, $F(s)G(s) = \int_0^\infty\int_0^t f(t-\tau)g(\tau)e^{-st}\,d\tau dt$

$$= \int_0^\infty\left(\int_0^t f(t-\tau)g(\tau)\,d\tau\right)e^{-st}\,dt = \int_0^\infty (f(t)*g(t))\,e^{-st}\,dt = \int_0^\infty h(t)e^{-st}\,dt = H(s).$$

(c) [6] $F(s) = \int_0^\infty f(t)e^{-st}\,d\tau = \int_0^1 f(t)e^{-st}\,d\tau + \int_1^\infty 0e^{-st}\,d\tau$. The improper integral converges for any s; the region of convergence is the whole complex plane. Continuing, $F(s) = \left.\frac{1}{-s}e^{-st}\right|_0^1 = \frac{1-e^{-s}}{s}$. [Why doesn't this blow up when $s \to 0$? The numerator goes to zero too, then, and the limit of the quotient (by l'Hopital for example) is 1.]

27. (a) [4] The Laplace transform of the equation $a\dot{w}+bw = \delta(t)$ is $asW(s)+bW(s) = 1$. Solve: $W(s) = \frac{1}{as+b} = \frac{1/a}{s+(b/a)}$ This is the Laplace transform of $w(t) = \frac{1}{a}u(t)e^{-bt/a}$.

(b) This is called the "unit ramp response."

(i) [6] $x_p = c_1t+c_0$, $ac_1+b(c_1t+c_0) = t$, $c_1 = \frac{1}{b}$ (as long as $b \neq 0$), $ac_1+bc_0 = 0$ so $c_0 = -\frac{a}{b^2}$, $x_p = \frac{1}{b}t - \frac{a}{b^2}$. $x(t) = x_p + ce^{-bt/a}$, so $0 = x(0) = -\frac{a}{b^2}+c$ and $x(t) = u(t)(\frac{1}{b}t - \frac{a}{b^2}(1-e^{-bt/a}))$. If $b = 0$ then $a\dot{x} = t$, which has general solution $x(t) = \frac{1}{2a}t^2 + c$. $0 = x(0) = c$, so $x(t) = u(t)\frac{1}{2a}t^2$.

(ii) [6] If $b \neq 0$: $w(t)*t = \int_0^t \frac{1}{a}e^{-b(t-\tau)/a}\tau\,d\tau = \frac{1}{a}e^{-bt/a}\int_0^t e^{b\tau/a}\tau\,d\tau$. Do this by parts:

$u=\tau$, $du = d\tau$, $dv = e^{b\tau/a}d\tau$, $v = \frac{a}{b}e^{b\tau/a}$, $w(t)*t = \dfrac{1}{a}e^{-bt/a}\left(\tau\dfrac{a}{b}e^{b\tau/a}\Big|_0^t - \displaystyle\int_0^t \dfrac{a}{b}e^{b\tau/a}\,d\tau\right) =$ $\frac{1}{a}e^{-tb/a}\left(t\frac{a}{b}e^{bt/a} - \frac{a^2}{b^2}(e^{bt/a}-1)\right) = \frac{1}{b}t - \frac{a}{b^2}(1-e^{-bt/a})$.

If $b=0$, $w(t)*t = \displaystyle\int_0^t \frac{1}{a}\tau\,d\tau = \frac{1}{a}\frac{t^2}{2}$.

(iii) [6] $a\dot{x}+bx = t$ has Laplace transform $asX+bX = \frac{1}{s^2}$, so $X = \dfrac{1}{s^2(as+b)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{as+b}$
Coverup: Multiply by s^2 and set $s=0$ to get $B = \frac{1}{b}$. Multiply by $as+b$ and set $s = -\frac{b}{a}$ to get $C = \frac{a^2}{b^2}$. Here's a clean way to get A: multiply through by s and then take s very large in size. You find $0 = A + \frac{C}{a}$, or $A = -\frac{a}{b^2}$. So $X = -\dfrac{a/b^2}{s} + \dfrac{1/b}{s^2} + \dfrac{a/b^2}{s+b/a}$, which is the Laplace transform of $x = -\frac{a}{b^2} + \frac{1}{b}t + \frac{a}{b^2}e^{-bt/a}$.

If $b=0$, $a\dot{x} = t$ has Laplace transform $asX = \dfrac{1}{s^2}$ so $X = \dfrac{1}{a}\dfrac{1}{s^3}$, and $x = u(t)\frac{1}{a}\frac{1}{2}t^2$

28. (a) [6] $w(t)$ has Laplace transform $W(s) = \dfrac{1}{3s^2+6s+6} = \dfrac{1}{3}\dfrac{1}{(s+1)^2+1}$. $\mathcal{L}[\sin t] = \frac{1}{s^2+1}$, so by s-shift $w(t) = \frac{1}{3}u(t)e^{-t}\sin t$.

(b) [14] $W(s) = \dfrac{1}{s^4+1}$. To use partial fractions we need to factor s^4+1, which is to say we need to find its roots. They are the fourth roots of -1, which are $r, \overline{r}, -r$, and $-\overline{r}$ where $r = \frac{1}{\sqrt{2}}(1+i)$. Now $\dfrac{1}{s^4+1} = \dfrac{1}{(s-r)(s-\overline{r})(s+r)(s+\overline{r})} = \dfrac{a}{s-r} + \dfrac{b}{s-\overline{r}} + \dfrac{c}{s+r} + \dfrac{d}{s+\overline{r}}$.
Coverup or cross-multiplication will lead to the coefficients. This is not pretty, and (per the web) I don't expect more.

[What I intended to ask was for the weight function for $D^4 - I$. Now the roots are ± 1 and $\pm i$, so we can write $\dfrac{1}{s^4-1} = \dfrac{a}{s-1} + \dfrac{b}{s+1} + \dfrac{c}{s-i} + \dfrac{d}{s+i}$. Coverup gives easily $a = b = \frac{1}{4}, c = \frac{i}{4}, d = -\frac{i}{4}$. So $w(t) = u(t)\frac{1}{4}\left(e^t + e^{-t} + ie^{it} - ie^{-it}\right) = u(t)\frac{1}{2}\left(\sinh(t) - \sin(t)\right)$.
I apologize for the mistake.]