18.03 Problem Set 5: Part II Solutions

Part I points: 17. 4, **18.** 0, **20.** 4, **21.** 4.

17. (a) [4] It seems that C must be close to 50 μF. The values of V_0 and R don't seem to matter.

(b) [10] Here is one of several ways to do this problem. We are looking at $L\ddot{I} + R\dot{I} + (1/C)I = V_0\omega\cos(\omega t)$. To undersand its sinusoidal solution, make the complex replacement $L\ddot{z} + I\dot{z} + (1/C)z = V_0\omega e^{i\omega t}$, so that $I_p = \mathrm{Re} z_p$. By the ERF, the exponential solution is $z_p = \frac{\omega e^{i\omega t}}{p(i\omega)}$. To be in phase with $\sin(\omega t)$, the real part of this must be a positive multiple of $\sin(\omega t)$. This occurs precisely when the real part of $p(i\omega)$ is zero. $\mathrm{Re}\, p(i\omega) = (1/C) - L\omega^2$, so the relation is $1/C = L\omega^2$.

To check, when $L = 500$ mH$=$ .5 H and $\omega = 200$ rad/sec, the system response is in phase when $C = 1/(.5 \times (200)^2) = 50 \times 10^{-6}$ F $= 50$ μF.

(c) [4] It seems that the maximal system response amplitude I_r occurs when $\omega = 100$ rad/sec, and that it is about 5 amps. Then the solution is in phase with the input voltage.

(d) [10] In **(b)** we saw that the solution is the real part of $z_p = \frac{\omega e^{i\omega t}}{p(i\omega)}$. The amplitude of this sinusoid is $\left|\frac{\omega}{p(i\omega)}\right|$, which is maximal when its reciprocal $\left|\frac{(1/C - L\omega^2) + Ri\omega}{\omega}\right| = \left|\left(\frac{1}{C\omega} - L\omega\right) + Ri\right|$ is minimal. The imaginary part here is constant, so as ω varies the complex number moves along the horizontal straight line with imaginary part R. The point on that line with minimal magnitude is Ri, which occurs when the real part is zero: $C/\omega = L\omega$, or $\omega_r = 1/\sqrt{LC}$. The amplitude is then $I_r = g(\omega_r)V_0 = V_0/R$. It depends only on V_0 and R, not on L or C! Finally, this is the same as the condition for phase lag zero, so the phase lag at $\omega = \omega_r$ is zero.

With the given values $R = 100\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, $\omega_r = 100$ rad/sec, as observed. When $V_0 = 500$ V and $R = 100\ \Omega$, $I_r = 5$ Amps, as observed.

18. [12] Notice that $\zeta^2 = \frac{b^2}{4m^2}\frac{m}{k} = \frac{b^2}{4mk}$, so $\omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \omega_n\sqrt{1 - \frac{b^2}{4mk}}$ or $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

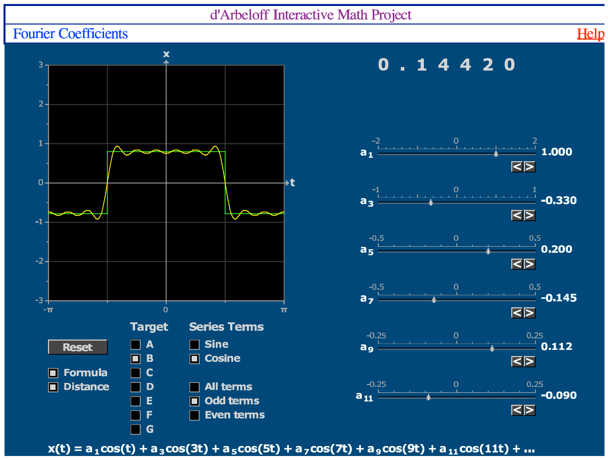
Solutions in the underdamped case have the form $x = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$. [From lecture: To see where the maxima are, notice that by the product rule for derivatives $\dot{x}$ is of the form $e^{-\zeta\omega_n t}$ times a sinusoid of circular frequency ω_d. It thus vanishes at times spaced by π/ω_d. Every other one is a maximum; they are spaced by $2\pi/\omega_d$.] Each time the peak is thus multiplied by a factor of $e^{-\zeta\omega_n(2\pi/\omega_d)} = e^{-2\pi\zeta/\sqrt{1-\zeta^2}}$. Thus after n cycles it is multiplied by a factor of $e^{-2\pi n\zeta/\sqrt{1-\zeta^2}}$ so $\frac{1}{2} = e^{-2\pi n\zeta/\sqrt{1-\zeta^2}}$ or $\frac{\alpha}{n} = \frac{\zeta}{\sqrt{1-\zeta^2}}$ where $\alpha = \frac{\ln 2}{2\pi} \simeq 0.1103178$. This solves out to $\zeta = \frac{\alpha/n}{\sqrt{1+(\alpha/n)^2}}$ When $n = 10$, $\frac{1}{\sqrt{1+(\alpha/n)^2}} \simeq 0.99993916$, so ζ is very close to $\alpha/10$.

20. (a) [2] Odd cosines work best; $a_1 = 1, a_3 = -\frac{1}{3}, a_5 = \frac{1}{5}, \ldots$.

(b) [8] $f(t)$ is even, so $b_n = 0$. For $n > 0$, $a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos(nt)\,dt =$
$\frac{2}{\pi}\left(\int_0^{\pi/2}\frac{\pi}{4}\cos(nt)\,dt + \int_{\pi/2}^{\pi}\frac{\pi}{4}(-\cos(nt))\,dt\right) = \frac{2}{\pi}\frac{\pi}{4}\left(\left[\frac{\sin(nt)}{n}\right]_0^{\pi/2} - \left[\frac{\sin(nt)}{n}\right]_{\pi/2}^{\pi}\right).$

Now $\sin(0) = \sin(n\pi) = 0$, and the upper limit of the first term coincides with the lower limit of the second, so $a_n = \frac{1}{n}\sin(\frac{n\pi}{2})$. When n is even these sine values are zero. The average value is 0, so $a_0 = 0$. When n is odd they alternate between $+1$ and -1. So the Fourier series is $f(t) = \cos(t) - \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) - \cdots$.

(c) [2]



21. (a) [4] The angle difference formula for sine gives $\sin(t - \frac{\pi}{3}) = -\sin(\frac{\pi}{3})\cos t + \cos(\frac{\pi}{3})\sin t = -\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t$ and this is the Fourier series. (If you don't remember the angle difference formula, you can use the complex exponential!: $\sin(t - \frac{\pi}{3}) = \mathrm{Im}(e^{i(t-\pi/3)}) = \mathrm{Im}(e^{-i\pi/3}e^{it}) = \mathrm{Im}((\frac{1}{2} - \frac{\sqrt{3}}{2}i)(\cos t + i\sin t)) = -\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t$.)

(b) [8] $\mathrm{sq}(t)$ is still odd, so $a_n = 0$, and, with $L = 2\pi$, $b_n = \frac{2}{2\pi}\int_0^{2\pi}\mathrm{sq}(t)\sin\left(\frac{nt}{2}\right)dt = \frac{1}{\pi}\left(\int_0^{\pi}\sin\left(\frac{nt}{2}\right)dt + \int_{\pi}^{2\pi}\left(-\sin\left(\frac{nt}{2}\right)\right)dt\right) = \frac{1}{\pi}\left(\left[-\frac{2}{n}\cos\left(\frac{nt}{2}\right)\right]_0^{\pi} - \left[-\frac{2}{n}\cos\left(\frac{nt}{2}\right)\right]_{\pi}^{2\pi}\right) = \frac{2}{\pi n}(-\cos(\frac{\pi n}{2}) + 1 + \cos(\frac{2\pi n}{2}) - \cos(\frac{\pi n}{2})) = \frac{2}{\pi n}c_n$, where $c_n = 1 - 2\cos(\frac{\pi n}{2}) + \cos(\pi n)$.

We evaluate c_n for some small values of n:

n	$\cos(\frac{\pi n}{2})$	$\cos(\pi n)$	c_n
0	1	1	0
1	0	-1	0
2	-1	1	4
3	0	-1	0

and then things repeat. So $b_n = 0$ unless $n = 2, 6, 10, \ldots$, and for such n, $b_n = \frac{8}{\pi n}$. The Fourier series is $\mathrm{sq}(t) = \frac{8}{\pi}(\frac{1}{2}\sin(\frac{2t}{2}) + \frac{1}{6}\sin(\frac{6t}{2}) + \cdots)$ *This is the same series as the Fourier series for* $\mathrm{sq}(t)$ *when it is regarded as having period* 2π. The numbering of the terms is different—only every fourth term is nonzero instead of every other term—but the series itself is identical.

(c) [8] $\mathrm{sq}(t - \frac{\pi}{4}) = \frac{4}{\pi}(\sin(t - \frac{\pi}{4}) + \frac{1}{3}\sin(3t - \frac{3\pi}{4})) + \cdots)$. Now $\sin(\theta - \phi) = (-\sin\phi)\cos\theta + (\cos\phi)\sin\theta$ and (with $\alpha = \sqrt{2}/2$)

$n =$	1	3	5	7
$-\sin(n\pi/4)$	$-\alpha$	$-\alpha$	α	α
$\cos(n\pi/4)$	α	$-\alpha$	$-\alpha$	α

so $\mathrm{sq}(t - \frac{\pi}{4}) = \frac{2\sqrt{2}}{\pi}((-\cos(t) - \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) + \frac{1}{7}\cos(7t) - - + + \cdots) + (\sin(t) - \frac{1}{3}\sin(3t) - \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + - - + \cdots))$.

(d) [4] $1 + 2\,\mathrm{sq}(2\pi t) = 1 + \frac{8}{\pi}(\sin(2\pi t) + \frac{1}{3}\sin(6\pi t) + \frac{1}{5}\sin(10\pi t) + \cdots)$.

(e) [4] $f(t) = \frac{\pi}{4}\mathrm{sq}(t + \frac{\pi}{2}) = \sin(t + \frac{\pi}{2}) + \frac{1}{3}\sin(3(t + \frac{\pi}{2})) + \cdots$. Now $\sin(\theta + \frac{\pi}{2}) = \cos\theta$ and $\sin(\theta + \frac{3\pi}{2}) = -\cos\theta$, so $f(t) = \cos t - \frac{1}{3}\cos(3t) + \frac{1}{5}\cos(5t) + \cdots$.

(f) [4] $g(t)$ is odd so it's given by a sine series. $g'(t) = \frac{4}{\pi}f(t)$, so the Fourier series of $g(t)$ is the integral of the Fourier series of $\frac{4}{\pi}f(t)$: $g(t) = \frac{4}{\pi}(\sin(t) - \frac{1}{3^2}\sin(3t) + \frac{1}{5^2}\sin(5t) - \cdots)$.

18.03 Differential Equations
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