18.03 Problem Set 3: Part II Solutions

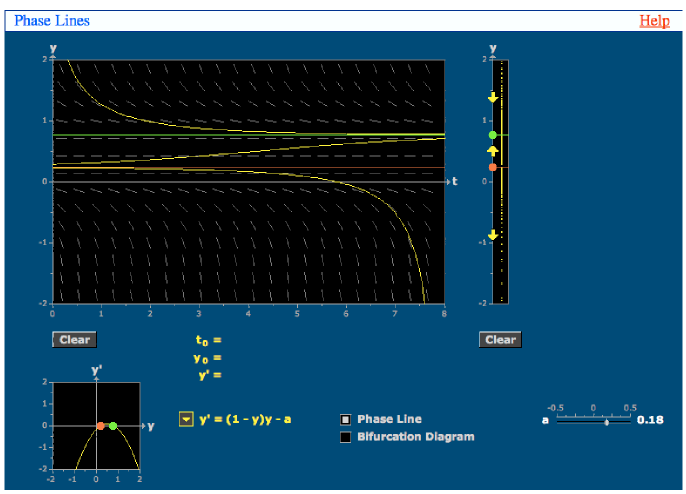
Part I points: 8. 8, **9.** 12, **11.** 9, **12.** 7.

8. (a) [3] The general logistic equation with small-population growth rate k_0 and equilibrium population p is $y = k_0(1-(y/p))y$, The top menu choice is $\dot{y} = (1-y)y - a$, which is the case $k_0 = 1$ and $p = 1$ together with a hunt rate of a. The only added assumption is $k_0 = 1$.

(b) [3] $0 = (1-y)y - a$ is the same as $y^2 - y + a = 0$, which by the quadratic formula has solutions $y = \frac{1}{2} \pm \sqrt{\frac{1}{4} - a}$. Thus when $a > \frac{1}{4}$ there are no equilibria; when $a = \frac{1}{4}$ there is one, namely $y_0 = \frac{1}{2}$, and it is semi-stable; and when $a < \frac{1}{4}$ there are two, the top one stable and the bottom one unstable.

(c) [3] 187.5 oryx is $\frac{3}{16}$ kilo-oryx, and $a = \frac{3}{16}$ leads to critical points $\frac{1}{2} \pm \frac{1}{4}$ or $\frac{1}{4}$ and $\frac{3}{4}$. So the stable equilibrium population is 750 animals, and the critical population below which it will crash is 250.

(d) [5]



(e) [2] $y^2 - y - a = 0$.

9. (a) [3] $y_0 = 3/4$, from **(b)** above: $y = u + \frac{3}{4}$, so $1 - y = \frac{1}{4} - u$ and $\dot{u} = (\frac{1}{4} - u)(u - \frac{3}{4}) - \frac{3}{16} = -\frac{1}{2}u - u^2$. No explicit time dependence, so autonomous; and if $u = 0$ then $\dot{u} = 0$.

(b) [3] The linearized equation is $\dot{u} = -\frac{1}{2}u$. The general solution to this is $u = ce^{-t/2}$.

(c) [3] Thus y is well approximated by $\frac{3}{4} + ce^{-t/2}$: the population decays, or relaxes, exponentially (with decay rate $\frac{1}{2}$) to the equilibrium value.

(d) [3] Both $p(t)$ and $q(t)$ must be constants.

11. **(a)** [4] $p(s) = \frac{1}{2}s^2 + \frac{3}{2}s + \frac{5}{8} = \frac{1}{2}(s^2 + 3s + \frac{5}{4})$. One way to find the roots is by completing the square: $s^2 + 3s + \frac{5}{4} = (s + \frac{3}{2})^2 - 1$, which clearly has roots $-\frac{3}{2} \pm 1$, or $-\frac{1}{2}$ and $-\frac{5}{2}$. This is what is shown on the applet.

(b) [4] $x = c_1 e^{-t/2} + c_2 e^{-5t/2}$. So $\dot{x} = -\frac{1}{2}c_1 e^{-t/2} - \frac{5}{2}c_2 e^{-5t/2}$, and $x_0 = c_1 + c_2$, $\dot{x}_0 = -\frac{1}{2}c_1 - \frac{5}{2}c_2$. Thus $x_0 + 2\dot{x}_0 = -4c_2$ so $c_2 = -\frac{1}{4}(x_0 + 2\dot{x}_0)$. Then $c_1 = x_0 - c_2 = \frac{1}{4}(5x_0 + 2\dot{x}_0)$.

(c) [3] x is purely exponential when either $c_1 = 0$—so $5x_0 + 2\dot{x}_0 = 0$—or when $c_2 = 0$—so $x_0 + 2\dot{x}_0 = 0$.

(d)[4] Try to solve for t in $0 = x(t) = c_1 e^{-t/2} + c_2 e^{-5t/2}$. This leads to $c_2/c_1 = -e^{2t}$. This admits a solution for some t exactly when c_1 and c_2 are of opposite sign. To get positive t, you need $c_2/c_1 < -1$: so either $-c_2 > c_1 > 0$ or $-c_2 < c_1 < 0$. In terms of $x_0, \dot{x}_0$, this says either $x_0 + 2\dot{x}_0 > 5x_0 + 2\dot{x}_0 > 0$, or $x_0 + 2\dot{x}_0 < 5x_0 + 2\dot{x}_0 < 0$, i.e. either $x_0 < 0$ and $\dot{x}_0 > \frac{5}{2}(-x_0)$, or $x_0 > 0$ and $\dot{x}_0 < -\frac{5}{2}x_0$. This is borne out by the applet.

12. **(a)** [6] $p(s) = \frac{1}{2}(s^2 + 2bs + \frac{5}{4}) = \frac{1}{2}((s + b)^2 + (\frac{5}{4} - b^2))$ has a double root when $\frac{5}{4} = b^2$ or $b = \frac{\sqrt{5}}{2}$. (We don't allow $b < 0$.) Then the root is $-b$, so the general solution is $(a + ct)e^{-bt}$.

(b) [6] When $b = \frac{1}{4}$, $p(s) = \frac{1}{2}(s^2 + \frac{1}{2}s + \frac{5}{4}) = \frac{1}{2}((s + \frac{1}{4})^2 + \frac{19}{16})$ has roots $-\frac{1}{4} \pm \frac{\sqrt{19}}{4}i \simeq -0.25 \pm (1.0897)i$. The general solution is thus $e^{-t/4}\left(a\cos\left(\frac{\sqrt{19}}{4}t\right) + b\sin\left(\frac{\sqrt{19}}{4}t\right)\right) = Ae^{-t/4}\cos\left(\frac{\sqrt{19}}{4}t - \phi\right)$. (Either form suffices.)

(c) [5] My measurements are: $0.00, 2.93, 5.76, 8.69, 11.52$. The successive differences are $2.93, 2.83, 2.93, 2.83$—pretty close to constant. This is half the period of the sinusoid involved in the solution, which has circular frequency $\omega = \frac{\sqrt{19}}{4}$ and hence half-period $\frac{\pi}{\omega} = \frac{4\pi}{\sqrt{19}} \simeq 2.8829231$. Not bad agreement! The oscillations are constant over time (though the amplitude decreases). Successive differences of zeros of other solutions should be the same.

18.03 Differential Equations
Spring 2010