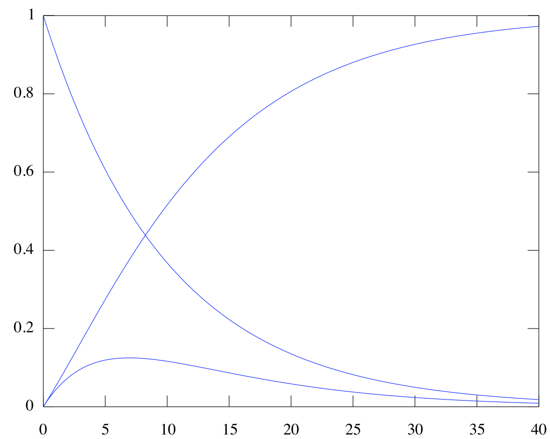
18.03 Problem Set 2: Part II Solutions

Part I points: 4. 4, **5.** 6 , **6.** 8, **7.** 6.

4. (a) [2] x falls; y rises and then falls; and z rises. With $\sigma = .1$, $\mu = .2$:



(b) [4] Startium obeys the natural decay equation, $\dot{x} = -\sigma x$, with solution $x = x(0)e^{-\sigma t}$. To relate σ to its half-life, solve for it in $x(0)/2 = x(0)e^{-\sigma t_S}$ to find $\sigma = (\ln 2)/t_S$. Similarly, $\mu = (\ln 2)/t_M$.

Midium decays as well, but in each small time interval gets half the decayed Startium added: so $y(t + \Delta t) \simeq -\mu y(t)\Delta t + \frac{1}{2}\sigma x(t)\Delta t$. Thus $\dot{y} = -\mu y + \frac{1}{2}\sigma x$. Endium receives half the decayed Startium and all the decayed Midium: $\dot{z} = \frac{1}{2}\sigma x + \mu y$. Adding these three equations gives $\dot{x} + \dot{y} + \dot{z} = 0$.

(c) [4] Using $x(0) = 1$, we know that $x = e^{-\sigma t}$. Thus $\dot{y} + \mu y = \frac{1}{2}\sigma e^{-\sigma t}$. An integrating factor is given by $e^{\mu t}$: $\frac{d}{dt}(e^{\mu t}y) = \frac{1}{2}\sigma e^{(\mu-\sigma)t}$. Integrating, $e^{\mu t}y = \frac{1}{2}\frac{\sigma}{\mu-\sigma}e^{(\mu-\sigma)t} + c$ or $y = \frac{1}{2}\frac{\sigma}{\mu-\sigma}e^{-\sigma t} + ce^{-\mu t}$. The initial condition is $y(0) = 0$, so $c = -\frac{1}{2}\frac{\sigma}{\mu-\sigma}$: $y = \frac{1}{2}\frac{\sigma}{\mu-\sigma}(e^{-\sigma t} - e^{-\mu t})$. We could solve for z in the same way, but it's easier to calculate $z = 1 - x - y = 1 + \frac{\sigma/2-\mu}{\mu-\sigma}e^{-\sigma t} + \frac{\sigma/2}{\mu-\sigma}e^{-\mu t}$

(d) [4] From the differential equation for y, we know that a critical point occurs when $\mu y = \frac{1}{2}\sigma e^{-\sigma t}$. Substitute the value for y: $\mu\frac{1}{2}\frac{\sigma}{\mu-\sigma}(e^{-\sigma t} - e^{-\mu t}) = \frac{1}{2}\sigma e^{-\sigma t}$. Some algebra leads to $\sigma e^{-\sigma t} = \mu e^{-\mu t}$, so $e^{(\mu-\sigma)t} = \mu/\sigma$, so $t_{\max} = \frac{\ln\mu - \ln\sigma}{\mu-\sigma}$.

(e) [2] Everything gets doubled.

(f) [4] If $x = e^t$ then $q(t) = t\dot{x} + 2x = te^t + 2e^t = (t+2)e^t$. The associated homogeneous equation is $t\dot{x} + 2x = 0$, which is separable: $dx/x = -2dt/t$, so $\ln|x| = -2\ln|t| + c = \ln(t^{-2}) + c$ and $x = C/t^2$. So the general solution of the original equation is $e^t + C/t^2$.

5. (a) [10] and **6. (a)** The rectangular expression gives the coordinates for the little pictures. Any angle may be altered by adding a multiple of 2π.

$1-i$	$\sqrt{2},\ -\pi/4$	$\sqrt{2}e^{-\pi i/4}$
$\sqrt{3}+i$	$2,\ \pi/6$	$2e^{\pi i/6}$
$(-1-i)/\sqrt{2}$	$1,\ 5\pi/4$	$e^{5\pi i/4}$
$(1+\sqrt{3}i)/2$	$1,\ \pi/3$	$e^{\pi i/3}$
$(-1+i)/\sqrt{2}$	$1,\ 3\pi/4$	$e^{3\pi i/4}$

(b) [8] (i) $\pm 1 \pm i$; or $\sqrt{2}e^{k\pi i/4}$ where $k = 1, 3, 5, 7$. (ii) $-1 \pm i$.

6. (a) [2] above.

(b) [3] $e^{a+bi} = e^a e^{bi}$ so $|e^{a+bi}| = |e^a||e^{bi}| = e^a$. Since $|-2| = 2$, $a = \ln 2$. $\text{Arg}(e^{a+bi}) = b$ up to adding multiples of 2π. $\text{Arg}(-1) = \pi$, so b is any odd multiple of π. Answer: $\ln 2 + b\pi i$, $b = \pm 1, \pm 3, \ldots$.

(c) [3] $\cos(4t) = \text{Re}e^{4it} = \text{Re}((e^{it})^4) = \text{Re}((\cos t + i \sin t)^4)$. By the binomial theorem, $(a+bi)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$, so we find $\cos(4t) = \cos^4 t - 6\cos^2 t \sin^2 t + \sin^4 t$.

(d) [8] (i) $w = 2\pi i$. The trajectory is the unit circle.

(ii) $w = -1$. The trajectory is the positive real axis.

(iii) $w = -1 + 2\pi i$. The trajectory is a spiral, spiralling in towards the origin in a counterclockwise direction and passing though 1.

(iv) $w = 0$. The trajectory is the single point 1.

7. (a) [8] $\dfrac{e^{3it}}{\sqrt{3}+i} = \dfrac{(\sqrt{3}-i)}{4}(\cos(3t) + i\sin(3t))$ has real part $\frac{\sqrt{3}}{4}\cos(3t) + \frac{1}{4}\sin(3t)$.

Form the right triangle with sides $a = \frac{\sqrt{3}}{4}$ and $b = \frac{1}{4}$. The hypotenuse is $A = 1/2$ and the angle is $\phi = \pi/6$.

$\sqrt{3}+i = 2e^{\pi i/6}$ (by essentially the same triangle), so $\dfrac{e^{3it}}{\sqrt{3}+i} = \dfrac{1}{2}e^{i(3t-\pi/6)}$: $B = \frac{1}{2}$, $\phi = \frac{\pi}{6}$, and $\text{Re}(Be^{i(3t-\phi)}) = B\cos(3t-\phi)$, so you get the same answer.

(b) [5] Substituting $z = we^{2it}$, $e^{2it} = w2ie^{2it} + 3we^{2it}$, so $1 = w(2i+3)$ or $w = \frac{1}{2i+3} = \frac{3-2i}{13}$. Thus a solution of the desired form is $z_p = \frac{3-2i}{13}e^{2it}$. The general solution is $z_p + ce^{-3t}$.

(c) [5] If $x = \text{Re}z$, the real part of $\dot{z} + 3z = e^{2it}$ is $\dot{x} + 3x = \cos(2t)$. Now $z_p = \frac{3-2i}{13}e^{2it} = \frac{3-2i}{13}(\cos(2t) + i\sin(2t))$ has real part $x_p = \frac{1}{13}(3\cos(2t) + 2\sin(2t))$. The general solution is then $x = x_p + ce^{-3t}$.